Solutions for In-Class Problems 13 for Wednesday, March 14

These problems are from Pre-Class Problems 13.

1. Find a polynomial p of degree 4 with zeros (roots) $\frac{2}{3}$ of multiplicity 2 and $-2i$ and $2i$ each of multiplicity 1.

2. Find a polynomial p of degree 3 with zeros (roots) $\frac{5}{3}$, $4+\sqrt{5}$ and $4-\sqrt{5}$ each of multiplicity 1.

3. Solve the following inequalities.

   a. $x^2 + 5x - 24 < 0$ b. $\dfrac{8-t}{4t+7} \le 0$ c. $\dfrac{x+2}{x^2-8x+16} > 0$

4. Determine the vertical asymptotes (if any).

   a. $f(x) = \dfrac{x^2-4}{2x^2-x-15}$ b. $g(x) = \dfrac{3x^2+2x-8}{x+2}$

SOLUTIONS:

1. Find a polynomial p of degree 4 with zeros (roots) $\frac{2}{3}$ of multiplicity 2 and $-2i$ and $2i$ each of multiplicity 1. Back to Problem 1.

In order for $\frac{2}{3}$ to be a zero (root) of multiplicity 2, $(3x-2)^2$ must be a factor of p.

In order for $-2i$ to be a zero (root) of multiplicity 1, $x+2i$ must be a factor of p.

In order for $2i$ to be a zero (root) of multiplicity 1, $x - 2i$ must be a factor of p.

Thus, $p(x) = a(3x - 2)^2(x + 2i)(x - 2i)$, where a is any nonzero real number.

We will use the special product formula $(a + b)(a - b) = a^2 - b^2$ for $(x + 2i)(x - 2i)$. Thus, $(x + 2i)(x - 2i) = x^2 - 4i^2$. Since $i = \sqrt{-1}$, then $i^2 = -1$. Thus, $(x + 2i)(x - 2i) = x^2 - 4i^2 = x^2 + 4$.

We will use the special product formula $(a - b)^2 = a^2 - 2ab + b^2$ for $(3x - 2)^2$. Thus, $(3x - 2)^2 = 9x^2 - 12x + 4$.

Thus, $p(x) = a(3x - 2)^2(x + 2i)(x - 2i) =$

$a(x^2 + 4)(9x^2 - 12x + 4) = a(9x^4 - 12x^3 + 4x^2 + 36x^2 - 48x + 16)$

$= a(9x^4 - 12x^3 + 40x^2 - 48x + 16)$

Thus, $p(x) = a(9x^4 - 12x^3 + 40x^2 - 48x + 16)$, where a is any nonzero real number.

If you pick a to equal one, then you get the polynomial
$p(x) = 9x^4 - 12x^3 + 40x^2 - 48x + 16$

Answer: $p(x) = 9x^4 - 12x^3 + 40x^2 - 48x + 16$

Back to Problem 1.

2. Find a polynomial p of degree 3 with zeros (roots) $\frac{5}{3}$, $4 + \sqrt{5}$ and $4 - \sqrt{5}$ each of multiplicity 1. Back to Problem 2.

In order for $\frac{5}{3}$ to be a zero (root) of multiplicity 1, $3x - 5$ must be a factor of p.

In order for $4 + \sqrt{5}$ to be a zero (root) of multiplicity 1, $x - (4 + \sqrt{5}) = x - 4 - \sqrt{5}$ must be a factor of p.

In order for $4 - \sqrt{5}$ to be a zero (root) of multiplicity 1, $x - (4 - \sqrt{5}) = x - 4 + \sqrt{5}$ must be a factor of p.

Thus, $p(x) = a(3x - 5)[x - 4 - \sqrt{5}][x - 4 + \sqrt{5}]$, where a is any nonzero real number.

We can easily find the product $[x - 4 - \sqrt{5}][x - 4 + \sqrt{5}]$ using the special product formulas $(a - b)(a + b) = a^2 - b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$.

First, use the special product formula $(a - b)(a + b) = a^2 - b^2$ with a being $x - 4$ and b being $\sqrt{5}$ and then use the second special product formula to find $(x - 4)^2$. Thus,

$[x - 4 - \sqrt{5}][x - 4 + \sqrt{5}] = (x - 4)^2 - (\sqrt{5})^2 =$
$x^2 - 8x + 16 - 5 = x^2 - 8x + 11$

Thus, $p(x) = a(3x - 5)[x - 4 - \sqrt{5}][x - 4 + \sqrt{5}] =$
$a(3x - 5)(x^2 - 8x + 11)$, where a is any nonzero real number.

Now, multiplying $(3x - 5)(x^2 - 8x + 11)$, we have that

$$(3x - 5)(x^2 - 8x + 11) = 3x^3 - 24x^2 + 33x - 5x^2 + 40x - 55 =$$

$$3x^3 - 29x^2 + 73x - 55.$$

Thus, $p(x) = a(3x - 5)(x^2 - 8x + 11) = 3x^3 - 29x^2 + 73x - 55$, where a is any nonzero real number.

If you pick a to equal one, then you get the polynomial $p(x) = 3x^3 - 29x^2 + 73x - 55$.

Answer: $p(x) = 3x^3 - 29x^2 + 73x - 55$ Back to Problem 2.

3a. $x^2 + 5x - 24 < 0$ Back to Problem 3.

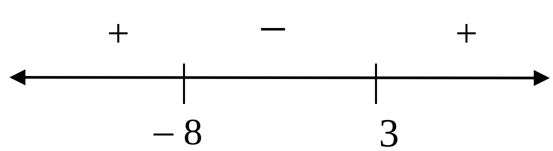
Step 1:

Find when the nonlinear expression $x^2 + 5x - 24$ is equal to zero. That is, solve the equation $x^2 + 5x - 24 = 0$.

$$x^2 + 5x - 24 = 0 \Rightarrow (x + 8)(x - 3) = 0 \Rightarrow x = -8,\ x = 3$$

Find when the nonlinear expression $x^2 + 5x - 24$ is undefined. The expression $x^2 + 5x - 24$ is defined for all real numbers x.

Step 2: Plot all the numbers found in Step 1 on the real number line.



Sign of $x^2 + 5x - 24$

Step 3: Use the real number line to identify the open intervals determined by the plotted numbers. Pick a test value for each open interval.

Interval	Test Value	Sign of $x^2 + 5x - 24 = (x + 8)(x - 3)$
$(-\infty, -8)$	-9	$(-9+8)(-9-3) = (-)(-) = +$
$(-8, 3)$	0	$(0+8)(0-3) = (+)(-) = -$
$(3, \infty)$	4	$(4+8)(4-3) = (+)(+) = +$

Answer: $(-8, 3)$

3b. $\dfrac{8 - t}{4t + 7} \leq 0$

Back to Problem 3.

NOTE: This is a **two** part problem. One part of the problem is to solve the nonlinear inequality $\dfrac{8 - t}{4t + 7} < 0$. The other part of the problem is to solve the equation $\dfrac{8 - t}{4t + 7} = 0$.

We will use the three step method to solve the nonlinear inequality $\dfrac{8 - t}{4t + 7} < 0$:

Step 1:

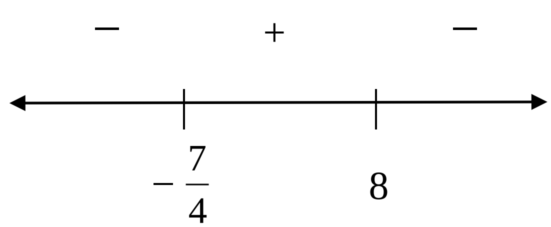
Find when the nonlinear expression $\dfrac{8 - t}{4t + 7}$ is equal to zero. That is, solve the equation $\dfrac{8 - t}{4t + 7} = 0$. The fraction is equal to zero if and only if the numerator of the fraction is equal to zero.

That is, $\dfrac{8 - t}{4t + 7} = 0 \Rightarrow 8 - t = 0 \Rightarrow t = 8$

Find when the nonlinear expression $\dfrac{8 - t}{4t + 7}$ is undefined. The fraction is undefined if and only if the denominator of the fraction is equal to zero.

That is, $\dfrac{8 - t}{4t + 7}$ undefined $\Rightarrow 4t + 7 = 0 \Rightarrow t = -\dfrac{7}{4}$

Step 2: Plot all the numbers found in Step 1 on the real number line.



Sign of $\dfrac{8 - t}{4t + 7}$

Step 3: Use the real number line to identify the open intervals determined by the plotted numbers. Pick a test value for each open interval.

Interval	Test Value	Sign of $\dfrac{8 - t}{4t + 7}$
$\left(-\infty, -\dfrac{7}{4}\right)$	-2	$\dfrac{8 + 2}{-8 + 7} = \dfrac{(+)}{(-)} = -$
$\left(-\dfrac{7}{4}, 8\right)$	0	$\dfrac{8 - 0}{0 + 7} = \dfrac{(+)}{(+)} = +$
$(8, \infty)$	9	$\dfrac{8 - 9}{36 + 7} = \dfrac{(-)}{(+)} = -$

Thus, the solution for the nonlinear inequality $\frac{8-t}{4t+7} < 0$ is the set of real numbers given by $\left(-\infty, -\frac{7}{4}\right) \cup (8, \infty)$. The solution for $\frac{8-t}{4t+7} = 0$ was found in Step 1 above. Thus, the solution for $\frac{8-t}{4t+7} = 0$ is the set $\{8\}$. Putting these two solutions together, we have that the solution for $\frac{8-t}{4t+7} \le 0$ is the set of real numbers $\left(-\infty, -\frac{7}{4}\right) \cup [8, \infty)$.

Answer: $\left(-\infty, -\frac{7}{4}\right) \cup [8, \infty)$

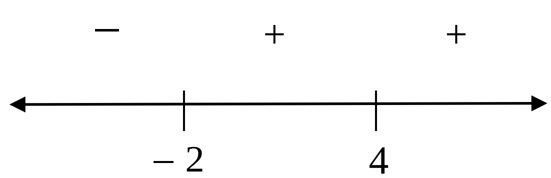
3c. $\frac{x+2}{x^2 - 8x + 16} > 0$ Back to Problem 3.

Step 1: $\frac{x+2}{x^2 - 8x + 16} = 0 \Rightarrow x + 2 = 0 \Rightarrow x = -2$

$\frac{x+2}{x^2 - 8x + 16}$ undefined $\Rightarrow x^2 - 8x + 16 = 0 \Rightarrow$

$(x-4)^2 = 0 \Rightarrow x - 4 = 0 \Rightarrow x = 4$

Step 2: Plot all the numbers found in Step 1 on the real number line.



Sign of $\frac{x+2}{x^2 - 8x + 16}$

Step 3: Use the real number line to identify the open intervals determined by the plotted numbers. Pick a test value for each open interval.

Interval	Test Value	Sign of $\dfrac{x+2}{x^2-8x+16} = \dfrac{x+2}{(x-4)^2}$
$(-\infty, -2)$	-3	$\dfrac{-3+2}{(-3-4)^2} = \dfrac{(-)}{(+)} = -$
$(-2, 4)$	0	$\dfrac{0+2}{(0-4)^2} = \dfrac{(+)}{(+)} = +$
$(4, \infty)$	5	$\dfrac{5+2}{(5-4)^2} = \dfrac{(+)}{(+)} = +$

Answer: $(-2,\ 4) \cup (4, \infty)$

Back to Problem 3.

4a. $f(x) = \dfrac{x^2 - 4}{2x^2 - x - 15}$

Back to Problem 4.

$$f(x) = \frac{(x+2)(x-2)}{(x-3)(2x+5)}$$

$$(x-3)(2x+5) = 0 \Rightarrow x = 3,\ x = -\frac{5}{2}$$

Answer: $x = -\dfrac{5}{2},\ x = 3$

4b. $g(x) = \dfrac{3x^2 + 2x - 8}{x + 2}$

Back to Problem 4.

$$g(x) = \frac{(x + 2)(3x - 4)}{x + 2} = 3x - 4, \; x \neq -2$$

The graph of the rational function g is the line $y = 3x - 4$ with the point $(-2, -10)$ missing. The rational function g does not have any vertical asymptotes.

Answer: None