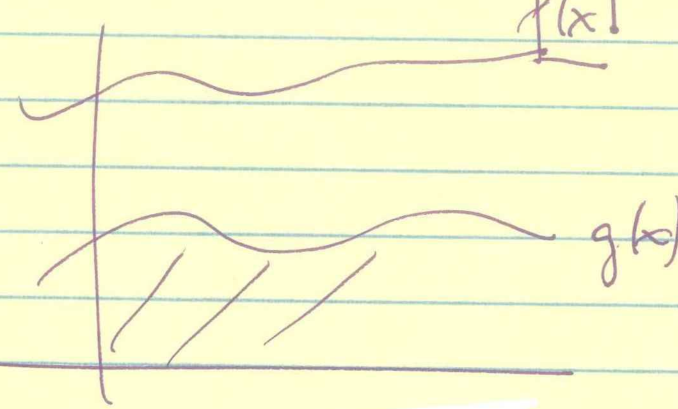
IMPROPER INTEGRALS, CONT'D:

COMPARISON THM:

Let $f(x), g(x) \geq 0$.

$\boxed{\text{Let } f(x) > g(x)}$



① If $\int_a^\infty g(x)\,dx$ is divergent,

then $\int_a^\infty f(x)\,dx$ is divergent.

② If $\int_a^\infty f(x)\,dx$ convergent, $\int_a^\infty g(x)\,dx$ is convergent.

EX1 Determine $\int_1^\infty e^{-x}\,dx$ is divergent or convergent.

$$\int_1^\infty e^{-x}\,dx = \lim_{t\to\infty} \int_1^t e^{-x}\,dx = \lim_{t\to\infty}\left[-e^{-x}\right]_1^t$$

$$= \lim_{t\to\infty} -\left[e^{-t} - e^{-1}\right] = \frac{1}{e}$$

So, this integral is convergent

EX2 Is $\int_1^{\infty} e^{-x^2}\,dx$ convergent?

We'll use comparison theorem.

Observe: $\frac{1}{e^x} > \frac{1}{e^{x^2}} \Rightarrow e^{-x} > e^{-x^2}$

so; by comparison theorem; since $\int_1^{\infty} e^{-x}\,dx$ is convergent, $\int_1^{\infty} e^{-x^2}\,dx$ is also convergent.

EX3 $\int_0^1 \frac{1}{1-x}$. Convergent or Divergent?

Improper int bec this is not defined at 1

Fact: (From last time) $\int_0^1 \frac{1}{x}\,dx$ is Divergent.

$x-1 < x$

Since $\frac{1}{x} \cancel{<} \frac{1}{x-1}$, $\int_0^1 \frac{1}{x-1}\,dx$ is also divergent by Comparison Thm.

IMPORTANT FAMILY: $y = \frac{1}{x^p}$, $p > 0$

When $p = 1$; $\int_1^\infty \frac{1}{x}\,dx$ is divergent.

When $p \neq 1$: $p > 1 \Rightarrow \int_1^\infty \frac{dx}{x^p} = \lim_{t\to\infty} \int_1^t \frac{1}{x^p}\,dx$

$$= \lim_{t\to\infty} \left[\frac{x^{-p+1}}{-p+1} \right]_1^t$$

$$= \lim_{t\to\infty} \left[\frac{x^{1-p}}{1-p} - \frac{x^0}{1-p} \right] = \frac{1}{p-1}.$$

$\to 0$ when $t \to \infty$.

So, this integral is Convergent

If $p < 1$; then $x^p < x \Rightarrow \frac{1}{x^p} > \frac{1}{x}$.

Since $\int_1^\infty \frac{dx}{x} < \int_1^\infty \frac{dx}{x^p}$

$\boxed{\text{Ex}}$

$$I = \int_e^\infty \frac{dx}{x(\ln x)^p} = \lim_{t\to\infty} \int_e^t \frac{dx}{x(\ln x)^p}$$

let $u = \ln x$
$\Rightarrow du = \frac{1}{x}\,dx$
$$\Rightarrow \int \frac{dx}{x(\ln x)^p} = \int \frac{du}{u^p} = \begin{cases} \ln|u|, & \text{if } p=1 \\ \dfrac{u^{-p+1}}{-p+1}, & \text{if } p\neq 1 \end{cases}$$

So,
$$I = \lim_{t\to\infty} \begin{cases} \ln|\ln t| - 0, & \text{if } p=1, \\ \dfrac{(\ln t)^{1-p}}{1-p} - \dfrac{(\ln e)^{1-p}}{1-p}, & \text{if } p\neq 1. \end{cases}$$

when $t\to\infty$
$\ln t \to \infty$

$$= \begin{cases} \infty, & \text{if } p=1. \\ \dfrac{1}{p-1}, & \text{if } p>1 \\ \infty, & \text{if } p<1 \end{cases}$$

So, I is divergent if $p\in(-\infty, 1]$
I is convergent if $p\in(1,\infty)$

Ex: $\int_2^\infty \frac{x^2+1}{x^5+x+2}\,dx$ Idea: Difference btw degrees is 3. So, compare it with $\frac{A}{x^3}$:

$x^5+x+2 > x^5 \Rightarrow \frac{1}{x^5} > \frac{1}{x^5+x+2}$ ①

Also; $x^2+1 \leq x^2+x^2$. ②

From ① & ②: $\frac{x^2+1}{x^5+x+2} < \frac{2x^2}{x^5} = \frac{2}{x^3}$

We know that $2\int_2^\infty \frac{1}{x^3}\,dx$ is convergent
($p=3>1$)

and, $\int_2^\infty \frac{x^2+1}{x^5+x+2}\,dx < \frac{2}{x^3}$

$\Rightarrow \int_2^\infty \frac{x^2+1}{x^5+x+2}\,dx$ is also convergent

Ex $I = \int_7^\infty \frac{x^2+1}{x^3+15x-2}\,dx$

<u>Idea</u>: Try to compare it with $\frac{1}{x}$.

$$\left.\begin{array}{l} x^2+1 > x^2 \\ x^3+\underbrace{15x-2} < x^3+x^3 \end{array}\right\} \Rightarrow \frac{x^2+1}{x^3+15x-2} > \frac{x^2}{2x^3} = \frac{1}{2x}$$

Since $\int \frac{1}{x}$ is divergent, I is also divergent

Ex $\int_1^\infty \frac{x^2+1}{e^{x^2}}\,dx$. <u>FACT</u>: $e^{ax} > x^N$ for x large and any a, N positive.

So, $e^{x^2} \geq x^{100}$ for x large

<u>Thus</u>, $\frac{x^2+1}{e^{x^2}} < \frac{2x^2}{x^{100}} = \frac{2}{x^{98}}$, $p = 98 > 1$

$$\Rightarrow \int_1^\infty \frac{x^2+1}{e^{x^2}}\,dx \text{ is CONVERGENT}$$

EX $I = \int_e^{\infty} \frac{\ln x}{\sqrt{x^3}}\,dx$ FACT: $\ln x < x^{1/N}$

$\frac{\ln x}{x^{3/2}} < \frac{x^{1/4}}{x^{3/2}}$ and $\int_e^{\infty} \frac{dx}{x^{5/4}}$ is convergent

(Bec: $p = 5/4 > 1$)

$\rightarrow$ I is convergent.

EX

$$\int_0^{\pi} \frac{\sin^2 x}{\sqrt{x}}\,dx \le \underbrace{\int_0^{\pi} \frac{1}{\sqrt{x}}\,dx}$$

is this convergent?

$$\int_0^{\pi} \frac{1}{\sqrt{x}}\,dx = \lim_{t \to 0^+} \int_t^{\pi} x^{-1/2}\,dx = \lim_{t \to 0^+} \left(2x^{1/2} \Big|_t^{\pi} \right)$$

not defined at 0

$$= \lim_{t \to 0^+} \left(2\sqrt{\pi} - \cancel{2\sqrt{t}} \right)$$

$$= 2\sqrt{\pi}$$

$\Rightarrow$ I is convergent by comparison theorem.