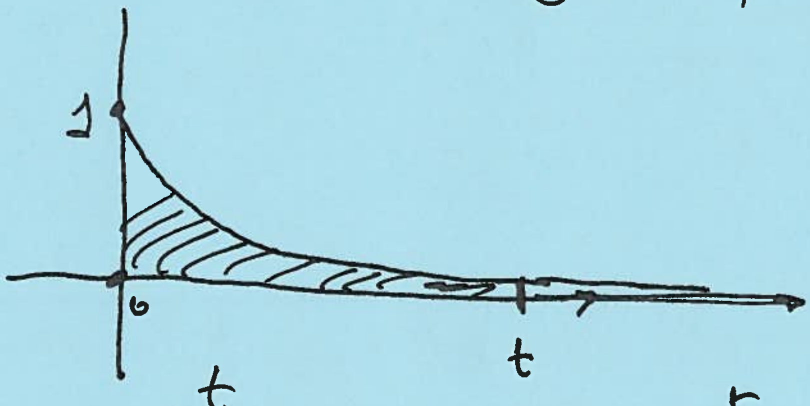
Lecture 6

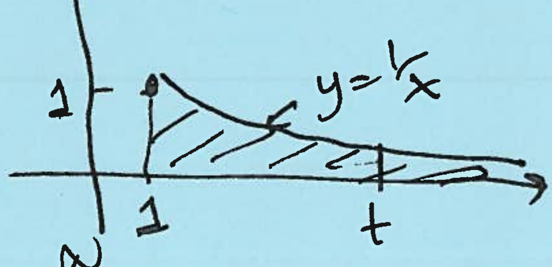
Question: what is the area under the graph of $y=e^{-x}$, $0\le x\le\infty$?

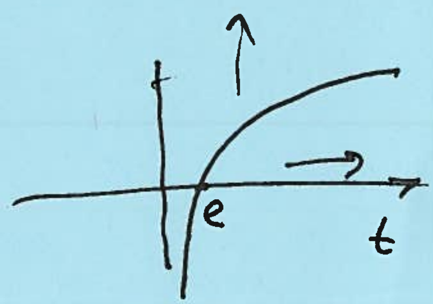


$$A(t)=\int_0^t e^{-x}\,dx = -e^{-x}\Big|_0^t = -e^{-t}+1$$

Define:
$$\text{Area}=\int_0^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_0^t e^{-x}\,dx = \lim_{t\to\infty}\left[1-\underbrace{e^{-t}}_{\to 0}\right] = 1$$

EX: What is the area under $y=\frac{1}{x}$, $1\le x<\infty$





$$A=\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty}(\ln t-0) = \infty$$

Divergent int.

Improper Integral

$e^{0.15} > e^{0.25}$

Type 1.

$$\int_a^\infty f(x)\,dx \qquad \int_{-\infty}^b f(x)\,dx \qquad \int_{-\infty}^\infty f(x)\,dx$$

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx \qquad \int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$

If limits exist $\Longleftrightarrow$ Integral Convergent

If limit d.n.e. (∞) $\Longrightarrow$ Integral is divergent.

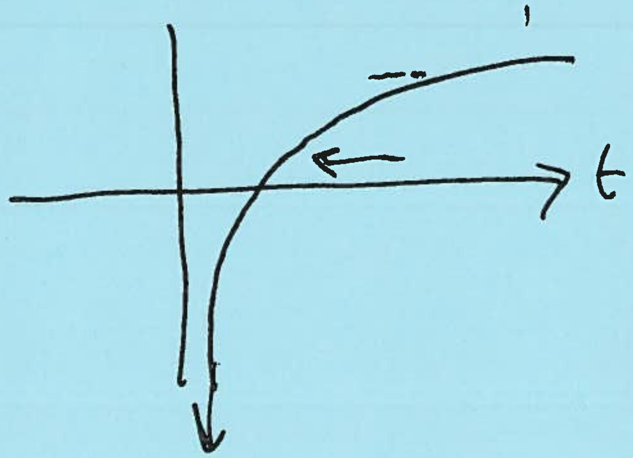
Type II

$$\int_0^1 \frac{1}{x}\,dx$$

Improper bec $f(x) = \frac{1}{x}$ is NOT defined at 0

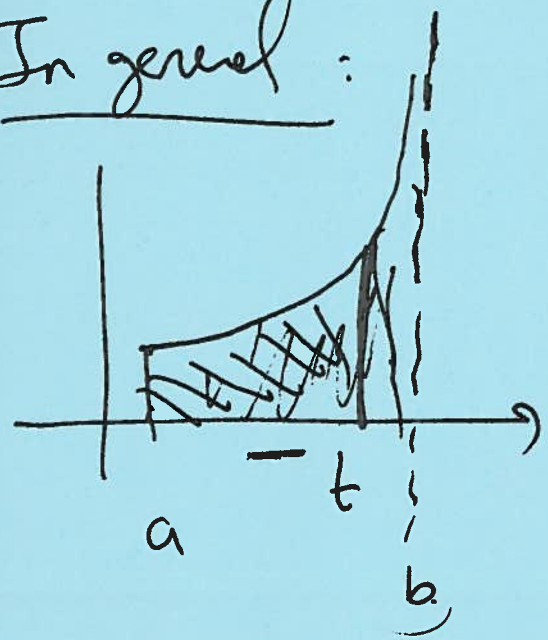
$$= \lim_{t\to 0^+}\int_t^1 \frac{1}{x}\,dx = \lim_{t\to 0^+}(\ln 1 - \ln t)$$

$$= \lim_{t\to 0^+}(-\ln t) = \infty$$



3

In general:



$$\int_a^b f(x)\,dx$$

$$\parallel$$

$$\lim_{t\to b^-}\int_a^t f(x)\,dt$$

IMPORTANT FAMILY $y=\frac{1}{x^p}$, $p>0$.

$$\int_1^\infty \frac{1}{x^p}\,dx$$

when $p=1$ $\int_1^\infty \frac{1}{x}\,dx$ is divergent.

when $p\neq 1$ $\begin{cases} \underline{p>1}: \end{cases}$ $\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^p}\,dx$

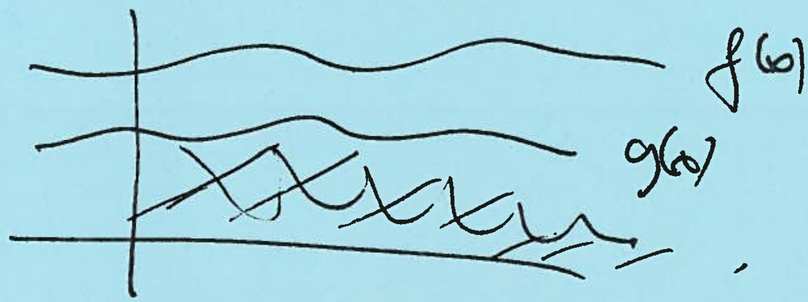
$$=\lim_{t\to\infty}\left[\frac{x^{-p+1}}{-p+1}\Big|_1^t\right]$$

$$=\lim_{t\to\infty}\left[\cancel{\frac{t^{1-p}}{1-p}}^{\,0} - \frac{1}{1-p}\right] = \frac{1}{p-1}$$

if $p<1$ $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ is divergent

Comparison Theorem: $f(x), g(x) \geq 0$

$f(x) > g(x)$



(1) If, $\displaystyle\int_a^\infty g(x)\,dx$ is divergent,

$\displaystyle\int_a^\infty f(x)\,dx$ is divergent

(2) If $\displaystyle\int_a^\infty f(x)$ is convergent $\displaystyle\int_a^\infty g(x)\,dx$ is convergent

Q: is $\displaystyle\int_1^\infty e^{-x^2}\,dx$ convergent?

Fact: $\displaystyle\int_1^\infty e^{-x}\,dx$ is convergent.

Compare: $e^{-x} = \dfrac{1}{e^{x}} \geq \dfrac{1}{e^{x^2}} = e^{-x^2}$

and: $\displaystyle\int_1^\infty e^{-x}\,dx$ convergent.

$\Big\}$ $\displaystyle\int_1^\infty e^{-x^2}\,dx$ is convergent.

(3) $\displaystyle\int_0^1 \frac{1}{x-1}\,dx$

~~Divergent~~

Fact: $\displaystyle\int_0^1 \frac{1}{x}\,dx$ is divergent.

$$\frac{1}{x-1} < \frac{1}{x}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

$$= \underbrace{\lim_{t\to-\infty}\int_t^0 f(x)\,dx}_{\mathrm{I}} + \underbrace{\lim_{t\to\infty}\int_0^t f(x)\,dx}_{\mathrm{II}}$$

convergent iff I & II are convergent.

Ex: $\int_0^1 \frac{1}{x}\,dx$ is divergent.

$$\int_0^1 \frac{1}{x^p}\,dx \qquad p > 1$$

$$\underbrace{\frac{1}{x}}_{g(x)} \not\geq \underbrace{\frac{1}{x^p}}_{f(x)}.$$