Last time: $\int \frac{P(x)}{Q(x)}\,dx$. $\deg P < \deg Q$

(otherwise: divide)

① Factor $Q(x)$

② Use the Partial fraction Theorem.

① $\dfrac{x^2+1}{\underbrace{x^3-x}_{x(x-1)\cdot(x+1)}} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{x+1}$

② $\dfrac{x^2+1}{x(x-1)^2\cdot(x+1)^3} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$

$+ \dfrac{D}{x+1} + \dfrac{E}{(x+1)^2} + \dfrac{F}{(x+1)^3}$

③ $\dfrac{2x^3+1}{x^3-x} = 2 + \dfrac{1+2x}{x^3-x} = 2 + \dfrac{1+2x}{x(x-1)(x+1)}$

$$\begin{array}{r|l} 2x^3+1 & x^3-x \\ \hline \mp 2x^3 \pm 2x & 2 \\ \hline 1+2x & \end{array}$$

$\rightarrow$ $2x^3+1 = 2.(x^3-x) + (1+2x)$

$\dfrac{2x^3+1}{x^3-x} = \dfrac{2.(x^3-x)+(1+2x)}{x^3-x}$

$= 2. + \dfrac{1+2x}{x^3-x}$

4) $$\frac{5x^2+2}{x.(x^2+2x+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+2}$$ 2

$$\Delta = 4-4.1.2 < 0$$

$$\left(\begin{array}{l} \text{In } ax^2+bx+c \\ \Delta = b^2-4ac \end{array} \right)$$

$$\frac{5x^2+2}{x(x^2+2x+2)} = \frac{A.(x^2+2x+2) + (Bx+C).x}{x.(x^2+2x+2)}$$

$$5x^2+2 = A(x^2+2x+2) + (Bx+C)x$$

$$\underline{5}x^2 + 2 = (\underline{A+B})x^2 + (2A+C)x + 2A$$

$$5 = A+B \qquad \boxed{B=4}$$

$$0 = 2A+C \Rightarrow \boxed{C=-2}$$

$$2 = 2A \Rightarrow \boxed{A=1}$$

$$\int \frac{5x^2+2}{x.(x^2+2x+2)}\,dx = \int \frac{1}{x}\,dx + \int \frac{4x-2}{x^2+2x+2}\,dx$$

$$= \ln x + \int \frac{4x+4-4-2}{x^2+2x+2}\,dx$$

$$= \ln x + \int \frac{4x+4}{x^2+2x+2}\,dx - \int \frac{6}{x^2+2x+2}\,dx$$

$x^2+2x+2=u \Rightarrow (2x+2)\,dx=du$ ③

$$I=\ln x+2.\ln(x^2+2x+2)-6\int\frac{dx}{\underbrace{x^2+2x+1}_{(x+1)^2}-1+2}$$

$$=\ln x+2.\ln(x^2+2x+2)-6.\int\frac{dx}{(x+1)^2+1}$$

$$=\ln x+2\ln(x^2+2x+2)-6.\arctan(x+1)\ +C$$

⑤ $$\frac{x^3+7x^2}{(x+1)(x^2+2)^2}=\frac{A}{x+1}+\frac{Bx+C}{x^2+2}+\frac{Dx+E}{(x^2+2)^2}$$

Strategy for integration

Substitution (

Trig. substitution Rationalizing Subs.

Ex $$\int\frac{\sqrt{x}}{x-1}\,dx$$

Let $u=\sqrt{x} \Longrightarrow du=\dfrac{1}{2\sqrt{x}}dx \Rightarrow (2u)du=dx$

$$I=\int\frac{u}{u^2-1}2u\,du=2.\int\frac{u^2-1+1}{u^2-1}du$$

$$I = 2\int \left[1 + \frac{1}{(u-1)(u+1)}\right] du$$

$$= 2\left[u + \int \frac{1/2}{u-1}du + \int \frac{-1/2}{u+1}du\right]$$

$$= 2u + 2\cdot\frac{1}{2}\cdot \ln|u-1| - 2\,\frac{1}{2}\cdot \ln|u+1| + C$$

$$= 2\sqrt{x} + \ln|\sqrt{x}-1| - \ln|\sqrt{x}+1| + C$$

$$= 2\sqrt{x} + \ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right| + C$$

$$\frac{1}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1}$$

$$= \frac{A(u+1)+B(u-1)}{(u-1)(u+1)}$$

$$1 = A(u+1) + B(u-1)$$

$A = 1/2$

$B = -1/2$

4

Ex $\int \arctan\left(\frac{1}{x}\right) dx$

$x = \cot\theta$

$dx = -\csc^2\theta\, d\theta$

1. $\int \underbrace{\arctan\left(\frac{1}{\cot\theta}\right)} (-\csc^2\theta)\, d\theta = -\int \theta\csc^2\theta\, d\theta$

2. $\arctan\left(\frac{1}{x}\right) = u$

$\Downarrow$

$$\frac{1}{1+\left(\frac{1}{x^2}\right)}\cdot\left(-\frac{1}{x^2}\right)dx = du$$

$$\frac{\cancel{x^2}}{1+x^2}\cdot\left(-\frac{1}{\cancel{x^2}}\right)dx = du$$

$dx = dv$

$\Downarrow$

$x = v$

$$I = x.\arctan\left(\frac{1}{x}\right) - \int \underbrace{x.\left(\frac{-1}{1+x^2}\right) dx}$$

$w = 1+x^2 \Rightarrow dw = 2x\,dx$

$$= x \arctan\left(\frac{1}{x}\right) + \frac{1}{2} \cdot \ln\left(1+x^2\right) + C$$

<u>**Ex**</u>: $\int \sin\left(\sqrt[3]{x}\right) dx = \ ?$