Lecture 4

①

TRIG. SUBSTITUTIONS

① $\sqrt{a^2 - x^2}$ use $x = a\sin\theta$ $\quad -\pi/2 \le \theta \le \pi/2$

② $\sqrt{a^2 + x^2}$ use $x = a\tan\theta \Rightarrow \sqrt{a^2+x^2} = a\sec\theta$

③ $\sqrt{x^2 - a^2}$ use: $x = a\sec\theta$ $\quad 0 \le \theta \le \pi/2$

$$\sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2(\underbrace{\sec^2\theta - 1}_{\tan^2\theta})} = a\tan\theta$$

Ex $\displaystyle\int \frac{\sqrt{9x^2-1}}{x^2}\,dx$

$$3x = \sec\theta \Rightarrow 3dx = \sec\theta\cdot\tan\theta\, d\theta$$

$$\sqrt{9x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta$$

$$I = \int \frac{\tan\theta}{\frac{\sec^2\theta}{9}}\,\frac{1}{3}\cdot\sec\theta\tan\theta\, d\theta$$

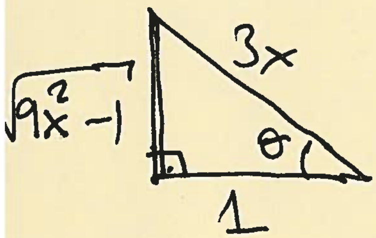
$$= 3\int \frac{\tan^2\theta}{\sec\theta}\,d\theta = 3\int \frac{\sec^2\theta - 1}{\sec\theta}\,d\theta$$

$$= 3\left[\int \sec\theta\, d\theta - \int \cos\theta\, d\theta\right]$$

$$= \overset{3}{\cancel{9}} \cdot \left[\ln|\sec\theta + \tan\theta| - \sin\theta \right] + C$$

$3x = \sec\theta$

$$= 3\left[\ln\left| 3x + \sqrt{9x^2-1} \right| - \frac{\sqrt{9x^2-1}}{3x} \right] + C$$



$\boxed{\text{EX2}}$ $\displaystyle\int \sqrt{x^2-4x+6}\,dx$

Complete x^2-4x to a square:

[In ax^2+bx+c, add & subtract $\left(\frac{b}{2a}\right)^2$]

$$x^2-4x+6 = x^2-4x+4-4+6 = (x-2)^2+2$$

let $u = x-2 \Rightarrow dx = du$

$$I = \int \sqrt{(x-2)^2+2}\,dx = \int \sqrt{u^2+2}\,du$$

let $u = \sqrt{2}\tan\theta$

$\sqrt{u^2+2} = \sqrt{2\tan^2\theta+2} = \sqrt{2}\sec\theta$

$du = \sqrt{2}\cdot\sec^2\theta\,d\theta$

$$I = 2\int \sec\theta\cdot\sec^2\theta\,d\theta$$

$$I = 2\,\frac{1}{2}\cdot\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right) + C$$ 3

$\tan\theta = \frac{u}{\sqrt{2}}$

$\Downarrow$

$\sec\theta = \sqrt{1+\frac{u^2}{2}}$

$$I = \sqrt{1+\frac{u^2}{2}}\cdot\frac{u}{\sqrt{2}} + \ln\left|\sqrt{1+\frac{u^2}{2}} + \frac{u}{\sqrt{2}}\right| + C$$

$$= \frac{\boxed{\sqrt{2+(x-2)^2}}\cdot(x-2)}{2} + \ln\left|\frac{\sqrt{2+(x-2)^2}}{\sqrt{2}} + \frac{(x-2)}{\sqrt{2}}\right| + C$$

7.4 RATIONAL FUNCTIONS

Integrate $R(x) = \frac{P(x)}{Q(x)}$

If $\deg Q(x) > \deg P(x)$: use Partial Fractions!

<u>EX</u>: $\int \frac{x^2+1}{x^3-x}\,dx$

Factor the denominator:

$$x^3 - x = x(x^2-1) = x(x-1)(x+1)$$

$$\frac{x^2+1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$(x-1)(x+1)$ $\quad x(x+1)$ $\quad (x-1)x$

Find A, B, C:

$$x^2+1 = A.(x-1).(x+1) + Bx.(x+1) + Cx(x-1)$$

For $x=1$,

$$2 = B.1.2 \Rightarrow \boxed{B=1}$$

for $x=-1$ $\quad 2 = C.(-1).(-2) \Rightarrow \boxed{C=1}$

for $x=0$ $\quad 1 = A(-1).1 \Rightarrow \boxed{A=-1}$

$$\int \frac{x^2+1}{x^3-x}\,dx = \int \frac{-1}{x}\,dx + \int \frac{dx}{x-1} + \int \frac{dx}{x+1}$$

$$= -\ln|x| + \ln|x-1| + \ln|x+1| + C$$

$$= \ln\left|\frac{(x-1)(x+1)}{x}\right| + C$$

<u>Ex</u>. $\displaystyle\int \frac{1}{x(x^2+1)}\,dx$

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} \Rightarrow 1 = A(x^2+1) + x.(Bx+C)$$

$$\Rightarrow 1 = (A+B).x^2 + Cx + A$$

$0.x^2 + 0.x$ $\quad A+B=0$

$C=0$

$A=1$

$$\int \frac{1}{x(x^2+1)}dx = \int \frac{1}{x}dx + \int \frac{-x}{x^2+1}dx$$

$u = x^2+1$
$\Downarrow$
$du = 2x\,dx$

$$= \ln|x| - \frac{1}{2}\int \frac{du}{u}$$

$$= \ln|x| - \frac{1}{2}\cdot \ln(x^2+1) + C$$

$\underline{\deg P \geq \deg Q}$

__Ex__: $\displaystyle\int \frac{x^3+1}{x^3-x} = \int \frac{\overbrace{x^3-x}+\overbrace{x+1}}{x^3-x}dx = \int \frac{x^3-x}{x^3-x} + \frac{x+1}{x^3-x}$

① Divide $\underline{x^3+1}$ to $\underline{x^3-x}$

$$\begin{array}{l|l} x^3+1 & x^3-x \\ \hline -x^3+x & 1 \\ \hline 1+x & \end{array}$$

Remainder.

$$= \int\left(1+\frac{x+1}{x^3-x}\right)dx$$

$$= \int dx + \underbrace{\int \frac{x+1}{x^3-x}dx}$$

Use Partial Fractions here.