Lecture 3 ①

Trigonometric integrals, Part II

$\sec x := \frac{1}{\cos x}$

$(\sec x)' = \sec x \cdot \tan x$

$(\tan x)' = \sec^2 x$

$1 + \tan^2 x = \sec^2 x$

$$\int \sec^m x \tan^n x \, dx \quad ?$$

① $\int \sec x \, dx = ?$

$$\int \sec x \, dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx = \int \frac{du}{u}$$

$u = \sec x + \tan x \Rightarrow du = \sec x \cdot \tan x + \sec^2 x$

$$= \ln|u| + C = \underline{\ln|\sec x + \tan x| + C}$$

② $\int \sec^3 x \, dx = \int \underbrace{\sec x}_{} \cdot \underbrace{\sec^2 x \, dx}_{}$

__IBP__: $u = \sec x \Rightarrow du = \sec x \cdot \tan x \, dx$

$\int \sec^2 x \, dx = \int dv \Rightarrow \tan x = v$.

$$I = \sec x \cdot \underline{\tan x} - \int \sec x \tan^2 x \, dx$$

$$= \sec x \tan x - \int \sec x \cdot (\sec^2 x - 1) \, dx$$

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x| + C'$$

$$\Rightarrow \int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \ln|\sec x + \tan x| + C$$

(3) $$\int \sec^3 x \tan^3 x\,dx = \int \sec^2 x \cdot \underbrace{\sec x \tan x}\, \tan^2 x\, \underline{dx}$$

$$u = \sec x \Rightarrow du = \sec x \cdot \tan x\,dx$$

$$I = \int u^2 (u^2 - 1)\,du = \int (u^4 - u^2)\,du = \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$$

(4) $$\int \sec x \cdot \underbrace{\tan^2 x}\,dx = \int \sec x \cdot (\sec^2 x - 1)\,dx$$

$$= \int \sec^3 x\,dx - \int \sec x\,dx$$

7.3 TRIGONOMETRIC SUBSTITUTION

EX: $I = \int \sqrt{4-x^2}\,dx$

Idea: Use trigonometric identities.

Let $\boxed{x = 2\sin\theta} \Longrightarrow dx = 2\cos\theta\,d\theta$

$$I = \int \sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\,d\theta = \int 2\sqrt{\cos^2\theta}\cdot 2\cos\theta\,d\theta$$

Check:

$$\sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta$$

Bec: $4-x^2 \ge 0 \Longrightarrow -2 \le x \le 2$.

$\Rightarrow -\pi/2 \le \theta \le \pi/2$.

$$I = 4\int \cos^2\theta\,d\theta = \cancel{4}^{2}\int \left[\frac{\cos 2\theta + 1}{\cancel{2}}\right] d\theta$$

$\cos 2\theta = 2\cos^2\theta - 1$

$$= 2\cdot\left[\frac{\sin 2\theta}{2} + \theta\right]$$

$$= 2\left[\frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}} + \theta\right]$$

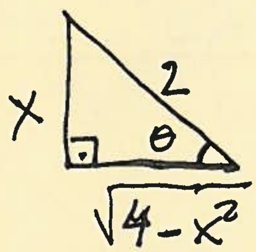
$x = 2\sin\theta$

$\Downarrow$

$\frac{x}{2} = \sin\theta$

(i) $\sin^2\theta + \cos^2\theta = 1 \implies \cos\theta = \sqrt{1 - \frac{x^2}{4}}$

(ii) Draw a right triangle:



$\frac{x}{2} = \sin\theta$

$\cos\theta = \frac{\sqrt{4-x^2}}{2}$

from the triangle

$$I = \frac{x}{2} \cdot \frac{\sqrt{4-x^2}}{2} + \arcsin\left(\frac{x}{2}\right) + C$$

$\boxed{Ex}$ $\int \frac{x^3}{\sqrt{9-x^2}}\,dx$

1. $x = 3\sin\theta$ $\quad \sqrt{9-x^2} = \sqrt{9(\cos^2\theta)} = 3\cos\theta$

$dx = 3\cos\theta$

$$I = \int \frac{27\sin^3\theta}{\cancel{3\cos\theta}} \cancel{3\cos\theta}\, d\theta = 27\int \sin^2\theta \sin\theta\, d\theta$$

$$= 27\int (1-\cos^2\theta)\sin\theta\, d\theta$$

$u = \cos\theta$ $\quad = -27 \cdot \int (1-u^2)\, du$

$du = -\sin\theta\, d\theta$ $\quad = -27\left(u - \frac{u^3}{3}\right) + C$

$$= -27\cdot\left(\cos\theta - \frac{\cos^3\theta}{3}\right) + C$$

$\frac{x}{3} = \sin\theta$

$\Downarrow$

$\cos^2\theta + \sin^2\theta = 1$

$\cos\theta = \sqrt{1 - \frac{x^2}{9}}$

$$= -27\left(\frac{\sqrt{9-x^2}}{3} - \frac{(9-x^2)^{3/2}}{3}\right) + C$$

(2) Substitution: $u = 9 - x^2$

$du = -2x\,dx$

$\boxed{EX}$ $\displaystyle\int \frac{dx}{\sqrt{16+x^2}}$

$1 + \tan^2\theta = \sec^2\theta$

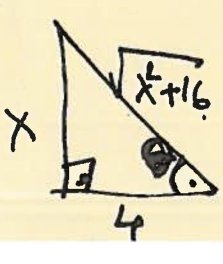
$\boxed{x = 4\tan\theta} \Rightarrow \sqrt{16 + 16\tan^2\theta} = 4\cdot\sqrt{1+\tan^2\theta}$

$= 4\sec\theta$

$\Downarrow$

$dx = 4\sec^2\theta\, d\theta$

$$I = \int \frac{4\sec^2\theta\, d\theta}{4\sec\theta} = \int \sec\theta\, d\theta = \ln\left|\sec\theta + \tan\theta\right| + C$$



$\frac{x}{4} = \tan\theta$

$$= \ln\left(\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right) + C$$

$\boxed{\text{Ex}}$ $\displaystyle\int \frac{x\,dx}{\sqrt{16+x^2}} = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \cdots$ (6)

$u = 16 + x^2 \Rightarrow du = 2x\,dx$

Ex $\displaystyle\int \frac{dx}{\sqrt{16+x^2}}$

Try: $x = 4\sin\theta$

$\sqrt{16+16\sin^2\theta} = 4\cdot\sqrt{1+\sin^2\theta}$

$\boxed{\cos^2\theta + \sin^2\theta = 1}$

$\tan^2\theta + 1 = \sec^2\theta$

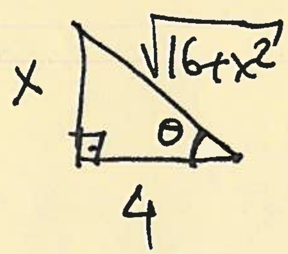
Let $x = 4\tan\theta \Rightarrow \sqrt{16+16\tan^2\theta} = 4\sec\theta$

$dx = 4\cdot\sec^2\theta\, d\theta$

$$I = \int \frac{4\sec^2\theta\, d\theta}{4\sec\theta} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left(\frac{\sqrt{16+x^2}}{4} + \frac{x}{4}\right) + C$$

$\frac{x}{4} = \tan\theta$



Ex: $\displaystyle\int \frac{x\,dx}{\sqrt{16+x^2}}$

Let $u = 16+x^2$

$\Downarrow$

$du = 2x\,dx$

$$= \frac{1}{2}\int \frac{du}{\sqrt{u}}$$