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TRIGONOMETRIC INTEGRALS

$\boxed{\text{Ex}}$ $\int \cos^2 x\, dx = ?$

WARNING: $\int x^2\, dx = \frac{x^3}{3} + C$

BUT $\int \cos^2 x\, dx \neq \frac{\cos^3 x}{3}$

So; we need to "get rid of" square.

Some useful identities:

① $\cos^2 x + \sin^2 x = 1$

② $\cos 2x = 2\cos^2 x - 1$
$= 1 - 2\sin^2 x$
$= \cos^2 x - \sin^2 x$.

③ $\sin 2x = 2 \sin x \cos x$

Using ②:

$\cos^2 x = \frac{\cos 2x + 1}{2}$.

so $I = \int \cos^2 x\, dx = \int \frac{\cos 2x + 1}{2}\, dx$

$= \frac{1}{2} \cdot \left[\int (\cos 2x + 1)\, dx\right]$

$= \frac{1}{2}\left[\frac{\sin 2x}{2} + \frac{x}{\cancel{1}}\right] + C$

$\boxed{\text{Ex}}$ $\int \cos^3 x\, dx = \int \cos x \cdot \underbrace{\cos^2 x}_{\text{use ①}}\, dx = \int \cos x (1 - \sin^2 x)\, dx$

$= \int \cos x\, dx - \int \underbrace{\cos x \cdot \sin^2 x}_{\text{substitute}}\, dx$

substitute $u = \sin x$, $du = \cos x\, dx$

$= \sin x - \left[\int u^2\, du\right]$

$= \sin x - \frac{\sin^3 x}{3} + C$

EX $\int \cos^2 x \sin^2 x \, dx = \int (\cos x \sin x)^2 dx$

use 3

$= \int \left(\frac{\sin 2x}{2}\right)^2$

$= \frac{1}{4}\left[\int \sin^2(2x)\, dx\right] = \frac{1}{4} \int \left(\frac{1-\cos 4x}{2}\right) dx$

use 1:

$\boxed{\cos 4x = 1 - 2\sin^2 2x}$

$= \frac{1}{8}\left[x - \frac{\sin 4x}{4}\right] + C$

EX

$\int \cos^2 x \sin^3 x \, dx = \int \cos^2 x \underline{\sin^2 x} \sin x \, dx$

2 EVEN, 3 ODD

$(1 - \cos^2 x)$

$= \int (\cos^2 x - \cos^4 x) \sin x \, dx$

$= \int \cos^2 x \sin x \, dx - \int \cos^4 x \sin x \, dx$

Substitute: $u = \cos x$
$du = -\sin x \, dx$

$= -\int u^2 du + \int u^4 du$

$= -\frac{u^3}{3} + \frac{u^5}{5} + C = \boxed{-\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C}$

We'll cover $\int \sec^m x \tan^n x \, dx$.

$\sec x = \frac{1}{\cos x}$

$(\sec x)' = \sec x \tan x$

$(\tan x)' = \sec^2 x$

$1 + \tan^2 x = \sec^2 x$

so,

$(\tan x)' = 1 + \tan^2 x$

1) $\int \sec x \, dx = ?$

$$\int \sec x \, dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx = \int \frac{du}{u}$$

Substitute: $\boxed{\begin{array}{l} u = \sec x + \tan x \\ \Rightarrow du = \sec x \tan x + \sec^2 x \end{array}}$

$$= \ln|u| + C = \ln|\sec x + \tan x| + C$$

2) $\int \sec^3 x \, dx = \int \underbrace{\sec x}_{u} \underbrace{\sec^2 x \, dx}_{dv}$.

IBP: $u = \sec x \Rightarrow du = \sec x \cdot \tan x \, dx$

$\int \sec^2 x \, dx = \int dv \Rightarrow \tan x = v$.

$$I = \sec x \cdot \tan x - \int \tan x (\sec x \tan x) \, dx$$

$\Rightarrow$

$$I = \sec x \tan x - \int \sec x \underbrace{(1 - \sec^2 x)}^{\tan^2 x} dx$$

$$\Rightarrow I = \sec x \tan x - \int \sec x \, dx - I$$

$$\Rightarrow 2I = \sec x \tan x - \ln|\sec x + \tan x| + C'$$

$$\Rightarrow \boxed{I = \frac{1}{2} \sec x \tan x - \frac{1}{2} \ln|\sec x + \tan x| + C}$$

3 $\int \sec^{3} x \tan^{3} x \, dx = \int \sec^2 x \sec x \tan x \underbrace{\tan^2 x}_{(\sec^2 x - 1)} dx$

ODD ODD

Substitute: $u = \sec x$
$\Rightarrow du = \sec x \tan x \, dx$

$$\Rightarrow I = \int u^2 (u^2 - 1) \, du = \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$$

4 $\int \sec x \tan^2 x \, dx = \int \sec x (\sec^2 x - 1) \, dx$

$$= \int \sec^3 x \, dx - \int \sec x \, dx$$

$$= \frac{1}{2} \sec x \tan x - \frac{1}{2} \ln|\sec x + \tan x| + C$$
$$- \ln|\sec x + \tan x|$$