Integrals involving Inverse Trig functions

Ex: $\displaystyle\int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3}\int \frac{du}{\sqrt{4-u^2}} = \frac{1}{3}\left[\sin^{-1}\left(\frac{u}{2}\right)+C\right]$

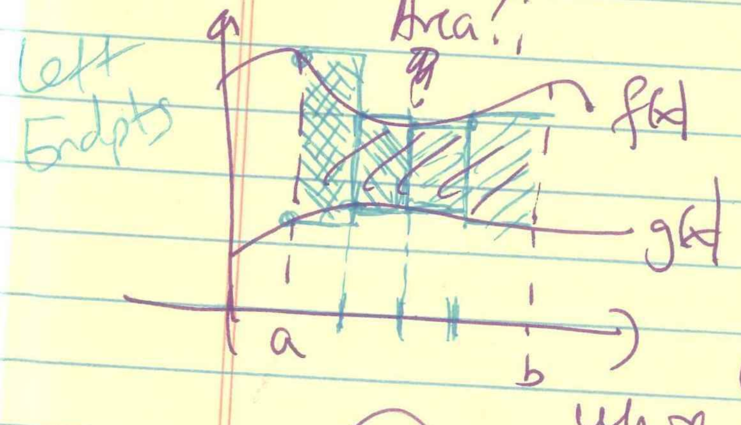
$u = 3x$
$du = 3\,dx$

Ex. $\displaystyle\int \frac{dx}{1+x^2} = \tan^{-1}x + C$

1/27

APPLICATIONS OF INTEGRATION

2.1 AREAS BTW CURVES:



We will do what we did before: APPROXIMATION.

① Divide [a,b] into equal small pieces (Δx)

② Fill the area with rectangles whose width is Δx.

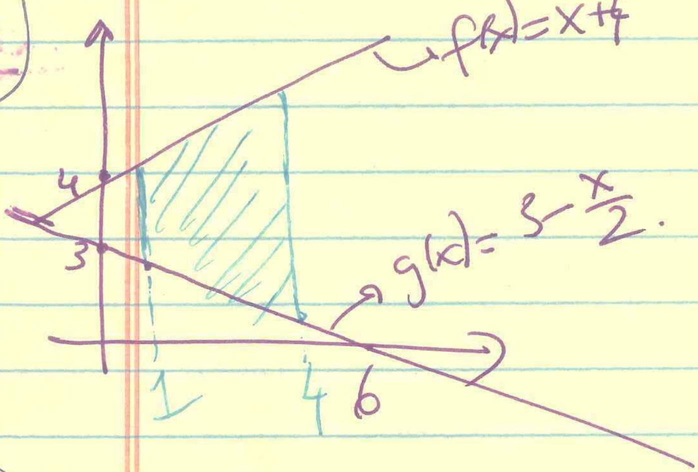
③ Find Area of each rectangle. Sum it to find Riemann Sum. (Appr area)

This time; Height $= f(x_i) - g(x_i)$

&:
$$\boxed{A \approx \underbrace{\sum_{i=0}^{n}\left[f(x_i) - g(x_i)\right]\Delta x}_{\text{Riemann Sum}}}$$

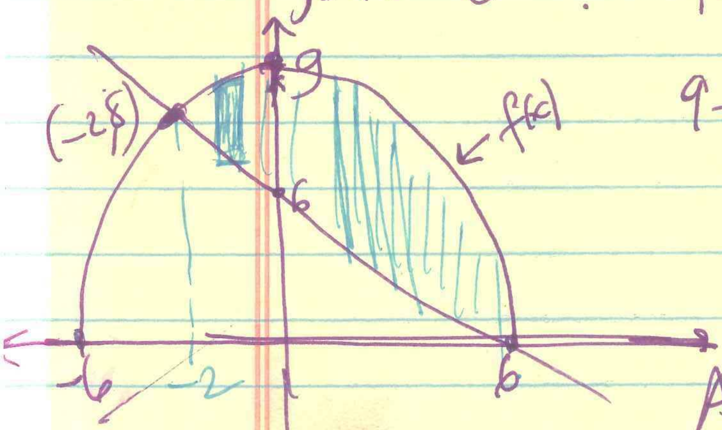
and Area $= \lim_{n\to\infty} \underbrace{\sum_{i=0}^{n}\left[f(x_i) - g(x_i)\right]\Delta x} = \int_a^b \left[f(x) - g(x)\right] dx$

Ex



$$A = \int_1^4 \left[x+4 - \left(3 - \frac{x}{2}\right)\right] dx = \frac{57}{4}$$

Ex R: Region bdd by $f(x) = 9 - \left(\frac{x}{2}\right)^2$ and $g(x) = 6 - x$. Find Area of R.



$$9 - \left(\frac{x}{2}\right)^2 = 6 - x \Rightarrow \frac{x^2}{4} - x - 3 = 0$$
$$\Rightarrow x^2 - 4x - 12 = 0$$
-6
$+2$

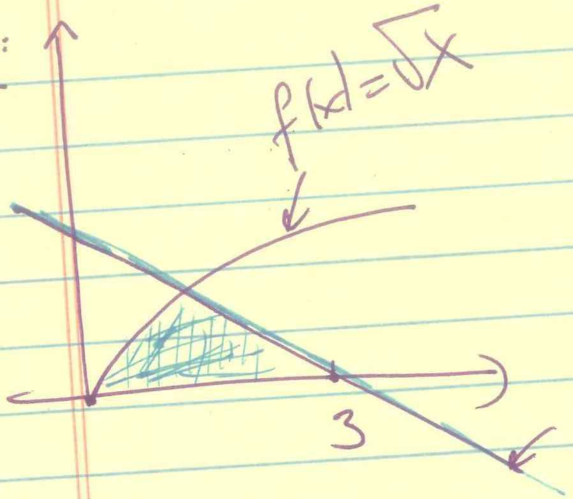
@ $x = -2$

$$A = \int_{-2}^{6} \left[\left(9 - \left(\frac{x}{2}\right)^2\right) - (6 - x)\right] dx$$

$\boxed{EX}$:

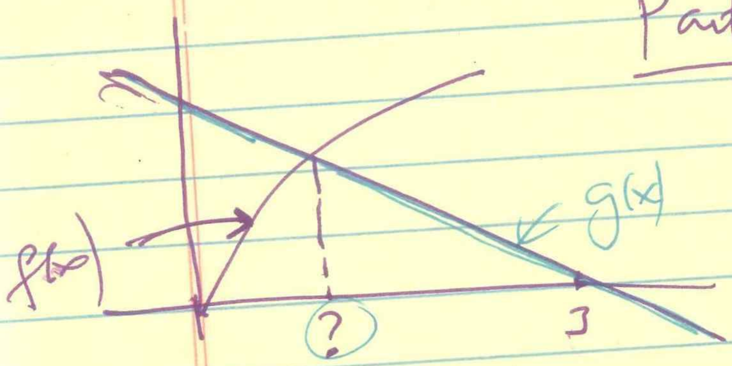
given R as on the left

Area $R = ?$



Observation: There are 2 areas!

We need to divide it into 2 parts:



Part 1: Upper curve: $f(x)=\sqrt{x}$
Lower curve $=$ x axis ($y=0$)

Part 2: Upper curve: $g(x)$
Lower curve: x axis

$f(x)$ and $g(x)$ intersect at:

Sol:

$g(x)=f(x) \Rightarrow \sqrt{x} = \frac{3}{2}-\frac{x}{2} \Rightarrow$

$x = \frac{9}{4}-\frac{3x}{2}+\frac{x^2}{4}$

$\Rightarrow x^2-10x+9=0$

$\Rightarrow (x-9)(x-1)$ So; $x=1$ is "?"

Now; $A = \int_0^1 [f(x) - 0]\,dx + \int_1^3 [g(x) - 0]\,dx$.

$$= \int_0^1 \sqrt{x}\,dx + \int_1^3 \left(\frac{3}{2} - \frac{x}{2}\right) dx.$$

$$= \frac{2}{3}\left(x^{3/2}\right)\Big|_0^1 + \left(\frac{3}{2}x - \frac{1}{2}\frac{x^2}{2}\right)\Big|_1^3$$

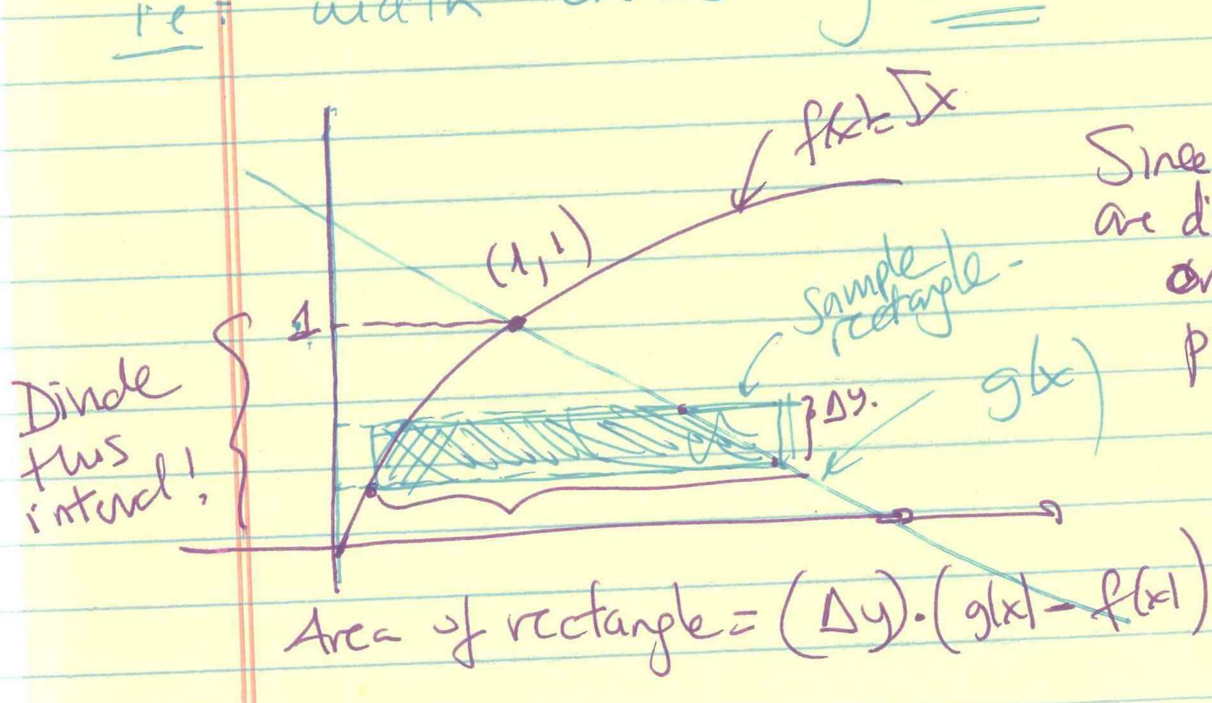
$$= \frac{2}{3} + \left[\left(\frac{9}{2} - \frac{9}{4}\right) - \left(\frac{3}{2} - \frac{1}{4}\right)\right] = \frac{2}{3} + \left[+\frac{9}{4} - \frac{5}{4}\right]$$

$$= 5/3.$$

Q: What Else can we do?

A: We can make our rectangles sit otherwise

ie: width on the y axis!



Since [0,1] we are dividing is now on y axis, pieces will be Δy!

Area of rectangle $= (\Delta y) \cdot (g(x) - f(x))$

But now we need to write $f(x) = \sqrt{x}$ and $g(x) = \frac{3}{2} - \frac{x}{2}$ in terms of y!

$y = \sqrt{x} \Rightarrow \boxed{x = y^2}$

$y = \frac{3}{2} - \frac{x}{2} \Rightarrow \frac{x}{2} = \frac{3}{2} - y \Rightarrow \boxed{x = 3 - 2y.}$

So; ~~Area~~ $= \int_0^1 \left[(3-2y) - y^2\right] dy.$

Check:

$$= \left(3y - \frac{2y^2}{2} - \frac{y^3}{3}\right)\Big|_0^1 = 3 - 1 - \frac{1}{3} = \frac{5}{3}. \checkmark$$