Approximation of $u_{xy} = \lambda(x, y)u$ by integrable PDEs

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Moutard equation

$$u_{xy} = \lambda(x, y)u \qquad (\mathcal{M})$$

has appeared first in Differential Geometry at the end of 19th century [Dar89, Bia27] and nowadays has numerous applications. For example in the theory of integrable 3-dimentional non-linear systems of PDE and in modern theory of solitons.

Given a partial solution $u = R$ of $(\mathcal{M})$ with some potential $\lambda = \lambda_0$, then for every additional partial solution $u = \phi$, there is a family of the corresponding solutions θ of $(\mathcal{M})$ with the potential $\lambda = \lambda_1$ defined by

$$\lambda_1 = R\left(\frac{1}{R}\right)_{xy}, \ (R\theta)_x = -R^2\left(\frac{\phi}{R}\right)_x, \ (R\theta)_y = R^2\left(\frac{\phi}{R}\right)_y .$$

Continuing in the same fashion we obtain a sequence of transformations:

$$\mathcal{M}_0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \dots ,$$

where $\mathcal{M}_i$ is the equation $(\mathcal{M})$ with the potential $\lambda = \lambda_i$.

Given $2k$ partial solutions of the initial equation $\mathcal{M}_0$, one can express the potential $\lambda = \lambda_k$ of the equation $\mathcal{M}_k$ and all its solutions [AN91]. The formula are analogous to the "wronskian" formula for the case of the Darboux transformations for 2-dimentional integrable PDEs [Dar89, TS09].

It has been also proved [Gan96] that the set of potentials obtainable from every fixed $\mathcal{M}_0$ is "locally dense" in the space of the smooth functions in the following sense. Let some potential λ_0 is defined in a neighborhood of $(0, 0)$, then for every $N \in \mathbb{N}_0$ there exists potential λ^* s.t. all its derivatives $D_J\lambda^*$, $|J| \leq N$ evaluated at $(0, 0)$ equal to any arbitrarily chosen numbers P_J:

$$D_J\lambda^*\big|_{(0,0)} = P_J ,$$

where $J = (x \dots xy \dots y)$, and $D_J = D_x \dots D_x D_y \dots D_y$.

In our talk we present our first results on the following problem: given an equation of the form $(\mathcal{M})$, find an approximation of its potential by the potential of some known to be integrable PDE.

References

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