**8.7 Improper Integrals:**

An improper integral of "type 1" is

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

provided the integrals and limit exist. Here $a$ is a constant. Alternatively

$$\int_{-\infty}^a f(x)\,dx = \lim_{t\to-\infty}\int_t^a f(x)\,dx.$$

Example: Evaluate $\int_2^\infty x^{-2}\,dx$.
Solution:

$$\int_2^\infty x^{-2}\,dx = \lim_{t\to\infty}\int_2^t x^{-2}\,dx = -\lim_{t\to\infty} x^{-1}|_2^t = -\lim_{t\to\infty} t^{-1} - 2^{-1} = 1/2$$

Picture. The area is 1/2!

Example: Evaluate $\int_2^\infty x^{-1}\,dx$. Solution:

$$\int_2^\infty x^{-1}\,dx = \lim_{t\to\infty}\int_2^t x^{-1}\,dx = \lim_{t\to\infty}\ln x|_2^t = \lim_{t\to\infty}\ln t - \ln 2$$

DNE! This area is infinite.

What do you guess $\int_2^\infty x^2\,dx$ is? $\int_2^\infty x^{-1/2}\,dx$?
Example: Evaluate $\int_2^\infty x^n\,dx$. if $n$ is some constant.
Solution: Assume $n \neq -1$

$$\int_2^\infty x^n\,dx = \lim_{t\to\infty}\frac{1}{n+1}x^{n+1}|_2^t = \lim_{t\to\infty}\frac{1}{n+1}[t^{n+1} - 2^{n+1}]$$

Need $n < -1$. And $n = 1$ won't do either by an earlier example.
Conclusion: If $a > 0$ then

$$\int_a^\infty \frac{1}{x^p}\,dx \text{ exists if and only if } p > 1$$

Picture the area.
Example: Evaluate $\int_0^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_0^t e^{-x}\,dx = \lim_{t\to\infty} 1 - e^{-t} = 1$
Comparison Test: Assume $0 \leq g(x) \leq f(x)$ for $x \geq a$. Then

1. If $\int_a^\infty f(x)\,dx$ converges then $\int_a^\infty g(x)\,dx$ exists.

2. If $\int_a^\infty g(x)\,dx$ diverges then $\int_a^\infty g(x)\,dx$ diverges.

Example: $\displaystyle\int_0^\infty \frac{\ln x}{x}\,dx = \infty$. Compare with $\displaystyle\int_0^\infty \frac{1}{x}\,dx = \infty$.
Example: $\displaystyle\int_0^\infty \frac{1}{x^2 \ln x}\,dx$ exists. Compare with $\displaystyle\int_0^\infty \frac{1}{x^2}\,dx$
Improper Integrals of Type 2. Functions discontinuous at $b$:

$$\int_a^b f(x)\,dx = \lim_{t\to b-} \int_a^t f(x)\,dx$$

Example: Evaluate (if it exists) $\int_0^3 x^{-1/2}\,dx$
Solution: The discontinuity is at 0!

$$\int_0^3 x^{-1/2}\,dx = \lim_{t\to 0+} \int_t^3 x^{-1/2}\,dx = \lim_{t\to 0+} 2x^{1/2}|_t^3 = \lim_{t\to 0+} 2\sqrt{3} - 2t^{1/2} = 2\sqrt{3}$$