8.4 Partial Fractions:

This is a method for evaluating integrals

$$\int \frac{p(x)}{q(x)}\,dx \text{ where } p(x), q(x) \text{ are polynomials}$$

Review: Irreducible Quadratics: x^2+x+1 does not factor. Criterion: $b^2-4ac<0$ means ax^2+bx+c does not factor, that is it's "irreducible." Recall the quadratic formula

$$r_\pm = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

tells how to factor quadratics: $ax^2+bx+c = a(x-r_+)(x-r_-)$. Every polynomial factors into linear factors $dx+e$ and irreducible quadratics.

Example: Evaluate the integrals

$$\int \frac{x^2}{x^2+1}\,dx$$
$$\int \frac{x^2+2x+1}{x^2-2x+1}\,dx$$
$$\int \frac{x^4-x+1}{x^3+x^2+x+1}\,dx$$

Step 1. Divide: If the degree on the top is equal or larger than the degree on bottom then divide.

$$\begin{aligned} \frac{x^2}{x^2+1} &= 1-\frac{1}{x^2+1} \\ \frac{x^2+2x+1}{x^2-2x+1} &= 1+\frac{4x}{x^2-2x+1} \\ \frac{x^4-x+1}{x^3+x^2+x+1} &= x-1+\frac{-x+2}{x^3+x^2+x+1} \end{aligned}$$

Step 2. Factor the bottom; x^2+1 is an irreducible quadratic; $x^2-2x+1=(x-1)^2$ and $x^3+x^2+x+1=(x+1)(x^2+1)$

Step 3. Expand by partial fractions. For example of $p(x)$ is a polynomial of degree at most _____ then

$$\frac{p(x)}{(x^2+1)^2(x-3)^3(x+1)} = \frac{Ax+B}{x^2+1}+\frac{Cx+D}{(x^2+1)^2}+\frac{E}{x-3}+\frac{F}{(x-3)^2}+\frac{G}{(x-3)^3}+\frac{H}{x+1}$$

for some constants $A, B, C, \ldots, H$.

Example

$$\frac{1}{x^2+1} = \frac{Ax+B}{x^2+1}$$

but this is nothing new: $A=0$, $B=1$. Consider

$$\frac{4x}{(x-1)^2} = \frac{A}{x-1}+\frac{B}{(x-1)^2}$$

and

$$\frac{-x+2}{(x+1)(x^2+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1}$$

Step 4: Solve for the constants A, B, C etc. We already did for the first example. For the second, multiply through by the denominator.

$$4x = A(x-1) + B = Ax + B - A$$

by gathering up the coefficients. Equate coefficients (Polynomials of degree n have at most n roots.)

$$\begin{aligned} A & & = 4 \\ -A & \; +B & = 0 \end{aligned}$$

So that $A = B = 4$ and

$$\frac{4x}{(x-1)^2} = \frac{4}{x-1} + \frac{4}{(x-1)^2}$$

In the third example, multiply by $(x+1)(x^2+1)$

$$-x+2 = (Ax+B)(x+1) + C(x^2+1) = (A+C)x^2 + (A+B)x + B + C$$

so that

$$\begin{array}{ccccccc} A & & & + & C & = & 0 \\ A & + & B & & & = & -1 \\ & & B & + & C & = & 2 \end{array}$$

Solve $A = -3/2 = -C$ $B = 1/2$.so that

$$\frac{-x+2}{(x+1)(x^2+1)} = \frac{1}{2}\frac{-3x+1}{x^2+1} + \frac{3/2}{x+1}$$

Step 5. Integrate.

$$\int \frac{x^2}{x^2+1}\,dx = \int 1 - \frac{1}{x^2+1}\,dx = x + \tan^{-1} x + C$$

and the next one is

$$\begin{aligned} \int \frac{x^2+2x+1}{x^2-2x+1}\,dx = \int 1 + \frac{4x}{(x-1)^2}\,dx &= \int 1 + \frac{4}{x-1} + \frac{4}{(x-1)^2}\,dx \\ &= x + 4\ln|x-1| - \frac{4}{x-1} + C \end{aligned}$$

Finally

$$\begin{aligned} \int \frac{x^4-x+1}{x^3+x^2+x+1}\,dx &= \int x - 1 + \frac{1}{2}\frac{-3x+1}{x^2+1} + \frac{3/2}{x+1}\,dx \\ &= \frac{x^2}{2} - x + \frac{-3}{4}\ln(x^2+1) + \frac{1}{2}\tan^{-1} x + \frac{3}{2}\ln|x+1| + C \end{aligned}$$