8.3 Trigonometric Substitution:

For Integrals Involving	Substitute	Use the Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta,\ dx = a\cos\theta\, d\theta$	$\sqrt{a^2 - x^2} = a\cos\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta,\ dx = a(\sec\theta)^2\, d\theta$	$\sqrt{a^2 + x^2} = a\sec\theta$

Remark: This is simply u-substitution with $u = \theta = \sin^{-1} x$ and $u = \theta = \tan^{-1} x$ so that $\pi/2 \le \theta \le \pi/2$.

Picture: If $x = a\sin\theta$ then

If $x = a\tan\theta$ then

Example: Evaluate $\int \frac{dx}{x^2\sqrt{9 - x^2}}$

Solution: Substitute $x = 3\sin\theta$, $dx = 3\cos\theta\, d\theta$. Then $\sqrt{9 - x^2} = \sqrt{9 - 9(\sin\theta)^2} = 3\cos\theta$ Therefore

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{9 - x^2}} &= \int \frac{3\cos\theta\, d\theta}{9(\sin\theta)^2 3\cos\theta} \\ &= \frac{1}{9}\int \frac{d\theta}{(\sin\theta)^2} \\ &= \frac{-1}{9}\cot\theta + C\end{aligned}$$

Now it is necessary to convert back into terms of x. Picture

so that $\cot\theta = (9 - x^2)^{1/2}/x$ and

$$\int \frac{dx}{x^2\sqrt{9 - x^2}} = \frac{-1}{9}\frac{(9 - x^2)^{1/2}}{x}$$

Check by differentiation:

$$\frac{-1}{9}\frac{d}{dx}\frac{(9-x^2)^{1/2}}{x} = \frac{-1}{9}\frac{x(9-x^2)^{-1/2}(-x)-(9-x^2)^{1/2}}{x^2} = \frac{-1}{9}(9-x^2)^{-1/2}\frac{-x^2-(9-x^2)}{9x^2}$$

and this checks.

Example: Evaluate $\int \frac{dx}{(25+x^2)^{3/2}}$

Solution: Substitute $x = 5\tan\theta$, so that $dx = 5(\sec\theta)^2$ and $(25+x^2)^{3/2} = (25+25(\tan\theta)^2)^{3/2} = 125(\sec\theta)^3$ so that

$$\begin{aligned}\int \frac{dx}{(25+x^2)^{3/2}} &= \int \frac{5(\sec\theta)^2}{125(\sec\theta)^3}\,d\theta \\ &= \frac{1}{25}\int \frac{1}{\sec\theta}\,d\theta \\ &= \frac{1}{25}\int \cos(\theta)\,d\theta \\ &= \frac{1}{25}\sin\theta + C\end{aligned}$$

Convert back into terms of x. Triangle:

$\sin\theta = x/(25+x^2)^{1/2}$ so that

$$\int \frac{dx}{(25+x^2)^{3/2}} = \frac{1}{25}\frac{x}{(25+x^2)^{1/2}} + C$$

Check by differentiation.

$$\frac{d}{dx}\frac{1}{25}\frac{x}{(25+x^2)^{1/2}} = \frac{1}{25}\frac{(25+x^2)^{1/2} - x\frac{1}{2}(25+x^2)^{-1/2}2x}{25+x^2} = \frac{1}{25}\frac{25+x^2-x^2}{(25+x^2)^{3/2}}$$

Example: Evaluate $\int \frac{x}{\sqrt{1-x^2}}\,dx$

Solution: Substitute $u = 1-x^2$, $du = -2x\,dx$ or $-\frac{1}{2}\,du = x\,dx$ so that

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int \frac{1}{u^{1/2}}\,du = -\frac{1}{2}2u^{1/2} + C = -(1-x^2)^{1/2} + C$$

Example: Evaluate $\int \frac{1}{\sqrt{9-x^2}}\,dx$

Solution: Factor out the "9".

$$\int \frac{1}{\sqrt{9-x^2}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{1-x^2/9}}\,dx$$

and then substitute $u = x/3$, $du = dx/3$ so that

$$\int \frac{1}{\sqrt{9-x^2}}\,dx = \int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1} u + C = \sin^{-1}(x/3) + C$$

Example:Evaluate $\int \sqrt{4-x^2}\,dx$

Solution: Here $x = 2\sin\theta$, $dx = 2\cos\theta$ and $\sqrt{4-x^2} = 2\cos\theta$ so that

$$\int \sqrt{4-x^2}\,dx = 4\int (\cos\theta)^2 d\theta$$

Use the trig identity $(\cos\theta)^2 = (1+\cos 2\theta)/2$:

$$\int \sqrt{4-x^2}\,dx = 2\int (1+\cos 2\theta)\,d\theta = 2\theta + \sin 2\theta + C = 2\theta + 2\sin\theta\cos\theta + C$$

Here the appropriate triangle is:

so that

$$\int \sqrt{4-x^2}\,dx = 2\sin^{-1}(x/2) + x\frac{(4-x^2)^{1/2}}{2} + C = 2\sin^{-1}(x/2) + \frac{1}{2}x(4-x^2)^{1/2} + C$$

Check by differentiation.

$$\begin{aligned}
\frac{d}{dx}[2\sin^{-1}(x/2) + \frac{1}{2}x(4-x^2)^{1/2}] &= \frac{1}{(1-(x/2)^2)^{1/2}} + \frac{1}{2}(4-x^2)^{1/2} + \frac{1}{2}x(4-x^2)^{-1/2}(-2x) \\
&= \frac{2}{(4-x^2)^{1/2}} + \frac{1}{2}\sqrt{4-x^2} - \frac{(1/2)x^2}{\sqrt{4-x^2}} = (4-x^2)^{1/2}
\end{aligned}$$