8.1 Integration by Parts:

Recall that the substitution method of integration is simply a method of reversing the chain rule of differentiation:

$$\frac{d}{dx}f(u(x)) = f'(u(x))u'(x)$$

Therefore if F has anti-derivative f ($f' = F$)

$$\int F(u(x))u'(x)\,dx = \int F(u)\,du = f(u) + C$$

Integration by parts inverts the product rule:

$$\frac{d}{dx}[u(x)v(x)] = u(x)\frac{d}{dx}v(x) + v(x)\frac{d}{dx}u(x)$$

Integrating this formula:

$$u(x)v(x) = \int u(x)\frac{d}{dx}v(x)\,dx + \int v(x)\frac{d}{dx}u(x)\,dx$$

Rearranging and using differential notation:

$$\int u\,dv = uv - \int v\,du$$

Example: Evaluate $\displaystyle\int x\sin x\,dx$.

Solution: Roughly the idea is to choose dv so that the antiderivative $v = \int dv$ is better or at least no worse and similarly the derivative u should be better than u: Try $u = x$ and $dv = \sin x\,dx$ so that $du = dx$ and $v = -\cos x$

$$\int x\sin x\,dx = -x\cos x - \int -\cos x\,dx = -x\cos x + \sin x + C$$

Check by differentiation: By the product rule

$$\frac{d}{dx}[-x\cos x + \sin x] = -x(-\sin x) - \cos x + \cos x = x\sin x.$$

Example: Evaluate $\displaystyle\int_0^\pi x\sin x\,dx$

Solution: Again $u = x$ and $dv = \sin x\,dx$ so that $du = dx$ and $v = -\cos x$:

$$\int_0^\pi x\sin x\,dx = -x\cos x|_0^\pi - \int_0^\pi -\cos x\,dx = -\pi\cos\pi - 0\cos 0 + \sin x|_0^\pi = \pi.$$

Example: Evaluate $\int x\ln x\,dx$.

Solution: Here $u =$ _____ $dv =$ _____dx so that $du = dx/x$ and $v = x^2/2$ (or $v = x^2/2 + C$!)

$$\int x\ln x\,dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\frac{dx}{x} = \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx = \frac{x^2}{2}\ln x + \frac{x^2}{4} + C$$

Check by differentiation.

Example: Evaluate $\int \ln x \, dx$.

Solution: Here $u = \ln x$ $dv = dx$ so that $du = dx/x$ and $v = x$ so that

$$\int \ln x \, dx = x \ln x - \int x \frac{dx}{x} = x \ln x - x + C$$

Check by differentiation.

Example: Evaluate $\int x^2 e^x \, dx$

Solution: Here $u = x^2$ and $dv = e^x \, dx$. Therefore $du = 2x \, dx$ and $v = e^x$

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx$$

Do it again. This time $u = x$ and $dv = e^x \, dx$ so that $du = dx$ and $v = e^x$ (and interchanging u and v goes backwards.)

$$\int x^2 e^x \, dx = x^2 e^x - 2[x e^x - \int e^x \, dx = x^2 e^x - 2x e^x + 2e^x + C$$

Again check by differetiation.

Example. Evaluate $\int \sin^{-1} x \, dx$

Solution: Let $u = \sin^{-1} x$ $dv = dx$. Then $du = \dfrac{dx}{(1-x^2)^{1/2}}$ and $v = x$

$$\int \sin^{-1} x \, dx = x \sin^{-1} x - \int \frac{x dx}{(1-x^2)^{1/2}}$$

Let $w = 1 - x^2$ so that $dw = -2x \, dx$

$$\int \sin^{-1} x \, dx = x \sin^{-1} x + \frac{1}{2} \int \frac{dw}{w^{1/2}} = x \sin^{-1} x + \frac{1}{2} 2w^{1/2} + C = x \sin^{-1} x + (1-x^2)^{1/2} + C$$

Example: Evaluate $\int e^x \cos x \, dx$

Solution: Here $u = \cos x$ and $dv = e^x \, dx$ (or vice versa). Then $du = -\sin x \, dx$ and $v = e^x$.

$$\int e^x \cos x \, dx = e^x \cos x - \int e^x (-\sin x) \, dx$$

Do it again: $u = \sin x$ and $dv = e^x \, dx$ so that $du = \sin x \, dx$ and $v = e^x$:

$$\int e^x \cos x \, dx = e^x \cos x + e^x \sin x - \int e^x \cos x \, dx$$

Solving for the unknown:

$$\int e^x \cos x \, dx = \frac{1}{2} e^x \cos x + \frac{1}{2} e^x \sin x + C$$

Remember the constant.