7.1 Natural Logarithm:
Recall:

- If $f(x) > 0$ then $\int_a^b f(x)\,dx$ is the area under the graph $y = f(x)$, $a \le x \le b$.
- $\displaystyle\int x^n dx =$ ________________ provided $n \ne -1$.
- The Fundamental Theorem of Calculus: Part 1: $\displaystyle\frac{d}{dx}\int_a^x f(t)\,dt =$ _____
- The Fundamental Theorem of Calculus: Part 2: $\displaystyle\int_a^b \frac{d}{dx} f(x)\,dx =$ __________

Definition: The *Natural Logarithm Function* $\ln x$ is defined for $x > 0$ by:

$$\ln x = \int_1^x \frac{1}{t}\,dt \text{ for } x > 0.$$

Therefore $\ln x$ is the area under the graph of $y = 1/t$ if $x > 1$ and negative that area if $x < 1$ and $\ln 1 =$ ___. Furthermore

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

Example Find the derivative of $\ln(3x + 1)$.

Remark: In Section 1.6 the textbook has introduced the $\ln x$ function as the inverse of the function e^x. We will see that the two definitions really are equivalent. The advantage of the defintion in 1.6 is that we can study the transcendental functions along with calculus: hence the “Early Transcendentals in the title of the text. The advantage of the present defintion is that is gives clear and coherent basis for studying all logarithm and exponential functions so that the properties can be most easily derived.

Properties of $\ln x$:

1. $\dfrac{d}{dx}\ln u = \dfrac{1}{u}\dfrac{du}{dx}$

2. $\ln(xy) = \ln x + \ln y$

3. $\ln(x/y) = \ln x - \ln y$

4. $\ln x^r = r\ln x$ if r is a fraction.

Verification of 1: Show $\ln(ab) = \ln a + \ln b$.

$$\begin{aligned}\ln(ab) = \int_1^{ab} \frac{1}{t}\,dt &= \int_1^a \frac{1}{t}\,dt + \int_a^{ab} \frac{1}{t}\,dt \\ &= \int_1^a \frac{1}{t}\,dt + \int_1^b \frac{1}{u}\,du \\ &= \ln a + \ln b\end{aligned}$$

where in the second integral on the right we substitute $u = t/a$, $du = dt/a$.

Verification of 2: See text.

Idea for 3: Special Case $r = 3$: $\ln(x^3) = 3\ln x$ for $\ln x^3 = \ln x + \ln x^2$ by 1 and so $\ln x^3 = \ln x + \ln x + \ln x$ by 1 again.

Second case $r = 1/4$: $\ln(x^{1/4}) = (\ln x)/4$ because $\ln y^4 = 4\ln y$ and choose $y = x^{1/4}$.

Example; Calculate $\dfrac{d}{dx}\ln|x|$

Solution:

$$\ln|x| = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$$

So if $x > 0$, $\dfrac{d}{dx}\ln|x| = \dfrac{d}{dx}\ln x = \dfrac{1}{x}$ but if $x < 0$, then $\dfrac{d}{dx}\ln|x| = \dfrac{d}{dx}\ln(-x) = \dfrac{1}{-x}(-1)$. Conclude

$$\frac{d}{dx}\ln|x| = \frac{1}{x}$$

Integration of Power Functions:

$$\int x^n\,dx = \begin{cases} \frac{1}{n+1}x^{n+1} + C & \text{if } n \neq -1 \\ \ln|x| + C & \text{if } n = -1 \end{cases}$$

Example: Compute $\displaystyle\int \frac{x^2}{x^3+1}\,dx$.

Solution: $\displaystyle\int \frac{x^2}{x^3+1} = \frac{1}{3}\ln|x^3+1| + C$

Example: Compute $\displaystyle\int \tan x\,dx$.

Solution: $\displaystyle\int \frac{\sin x}{\cos x}\,dx = -\int \frac{1}{u}\,du$ where $u = \cos x$ so that $du = \sin x dx$. This last expression is $-\ln|u| + C$.

$$\int \tan x\,dx = -\ln|\cos x| + C = \ln|\sec x| + C$$

Check by differentiation: $\frac{d}{dx}(-\ln|\cos x|) = \tan x$.

Similarly

$$\int \cot x\, dx = \ln|\sin x| + C.$$

More Properties of $\ln x$:

- $\ln x$ is increasing for $x > 0$ because $\frac{d}{dx}\ln x = \frac{1}{x}$. So $\ln x$ is one-to-one.
- $\lim_{x\to\infty} \ln x = \infty$ because $\ln 2^r = r\ln 2$ and let $r \to \infty$.
- $\lim_{x\to 0} \ln x = -\infty$ because $\ln(1/x) = -\ln x$.

It follows that $\ln x$ is an increasing function $d/dx(\ln x) = 1/x > 0$ (and so is "1–1") that maps the open interval $(0, \infty)$ to the entire real line.

Definition: We introduce the number e defined to be the unique number so that $\ln e = 1$ ($\ln x$ takes on every real value once and only once).

$$e = 2.718281828459\ldots$$

By the properites of logarithms, for any rational number r

$$\ln[e^r] = r\ln e = r$$

We can now proceed just as the text did in Chapter 1. Because $\ln x$ is 1–1 it has an inverse function which we call $E(x)$, for the moment. The $E(\ln x) = x$ and $\ln(E(x)) = x$. The latter property assures that $E(r) = e^r$ for every rational exponent r. However $E(x)$ is defined for all x and so we define $e^x = E(x)$

Definition: Define the natural exponential $\exp(x)$ to be the inverse of $\ln x$.

$$\exp(\ln x) = x; \qquad\qquad \ln(\exp x) = x$$

With $x = 1$, $\exp(0) = 1$ and with $x = e$, $\exp(1) = e$ and with $x = e^3$, $\exp(3) = e^3$. In general $\exp(r) = e^r$ for r rational and we write $\exp(x) = e^x$ for all real x.

Graph:

Properties:

1. $e^{\ln x} =$ _____
2. $\ln e^x =$ _____
3. $e^0 =$ __
4. $e^{x+y} = e^x e^y$
5. $e^{-x} = \dfrac{1}{e^x}$
6. $(e^x)^r = e^{rx}$
7. $\dfrac{d}{dx} e^x =$ _____
8. $e^x > 0$ for all x
9. $\lim_{x\to\infty} e^x = \infty$
10. $\lim_{x\to-\infty} e^x = 0$

Verification of 4. Since ln is a one-to-one function it suffices to show that $\ln(e^{x+y}) = \ln(e^x e^y)$. But $\ln(e^{x+y}) = x + y$ by 2. And $\ln(e^x e^y) = \ln(e^x) + \ln(e^y)$ by one of the properties of ln. Applying Property 2, $\ln(e^x) + \ln(e^y) = x + y$. Verification of 7. We know that e^x is differentiable because its inverse $\ln x$ is differentiable with derivative $1/x$ which is nonzero. Differentiate in 2.

$$\begin{aligned} \frac{d}{dx}\ln(e^x) &= \frac{d}{dx}x \\ \frac{1}{e^x}\frac{d}{dx}e^x &= 1 \end{aligned}$$

so that

$$\frac{d}{dx}e^x = e^x$$

or incorporating the chain rule:

$$\frac{d}{dx}e^u = e^u\frac{du}{dx}.$$

Example: If $f(x) = e^{x^2+3x}$ then $f'(x) = (2x+3)e^{x^2+3x}$

Example: Evaluate $\int e^{x^2} x\,dx$. Substitute $u = x^2$ ... $\int e^{x^2} x\,dx = (1/2)ex^2 + C$

General Logarithms and Exponential:

Observe that $2 = e^{\ln 2}$ so that for any rational x $2^x = e^{x\ln 2}$.

Definition: For $a > 0$ $a^x = e^{x\ln a}$.

For example $2^x \sim e^{(0.69314718)x}$. Therefore

$$\frac{d}{dx}a^x = (\ln a)a^x$$

Example: $\dfrac{d}{dx}10^x = (\ln 10)10^x \sim (2.3025851)10^x$.
Example: $\dfrac{d}{dx}10^{\sin x} = (\ln 10)10^{\sin x}\cos x$.
Example: $\dfrac{d}{dx}[e^x + x^e] = e^x + ex^{e-1}$
Graphs: Graph 2^x, e^x and 10^x on one set of axes.

Example: Evaluate $\int 2^{x^2+6x}(x+3)\,dx$. Substitute $u = x^2+6x$ so that $du = (2x+6)dx$.
Definition: $log_a x$ is the inverse a^x:

$$a^{\log_a x} = x \qquad\qquad log_a(a^x) = x$$

Remark: $\log_a x = \dfrac{\ln x}{\ln a}$. In particular $\log_{10} x \sim \dfrac{\ln x}{2.3025851}$.
Examples: Evaluate $\displaystyle\int \frac{\log_2 x}{x}\,dx = \frac{1}{\ln 2}\int \frac{\ln x}{x}\,dx$ Substitute $u = \ln x$.

$$\int \frac{\log_2 x}{x}\,dx = \frac{1}{2\ln 2}(\ln x)^2 + C$$