6.3 Areas of Surfaces of Revolution:

Consider the curve $y = f(x)$, $a \leq x \leq b$, in the case $f(x) \geq 0$. Rotate the curve about the x-axis and it traces out a surface. What is the area of that surface? In other words if the surface is painted then how much paint is needed? Look at a tiny piece of the curve. It has length $\sqrt{1+(f'(x))^2}\,dx$. It traces out the frustrum of a cone (roughly) and it has area $2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$. The surface area is

$$\int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$

Example: The curve $y = \sqrt{x+3}$, $3 \leq x \leq 9$ is rotated about the x-axis. Find the area or the surface generated.

Compute $f'(x) = (1/2)(x+3)^{-1/2}$ so that the area is

$$\begin{aligned} SA &= \int_3^9 2\pi\sqrt{x+3}\sqrt{1+((1/2)(x+3)^{-1/2})^2}\,dx \\ &= \int_3^9 2\pi\sqrt{(x+3)(1+\frac{1}{4(x+3)}}\,dx \\ &= 2\pi\int_3^9 (x+3+(1/4))^{1/2}\,dx \\ &= 2\pi\frac{2}{3}(x+3+(1/4))^{3/2}|_3^9 = \frac{4\pi}{3}[(49)^{3/2}/8-(25)^{3/2}/8] = \frac{109\pi}{3} \end{aligned}$$

If the curve is $x = g(y)$, $c \leq y \leq d$ is rotated about the y axis then the surface generated has area

$$\int_c^d 2\pi g(y)\sqrt{1+(g'(y))^2}\,dy$$