6.3 Arc Length:

Find the length of the curve $y = f(x)$, $a \le x \le b$. Look at the graph. The distance from $(x_1, f(x_1))$ an arbitrary point on the graph to $(x_1 + h, f(x_1 + h))$ is

$$((\Delta x)^2 + (f(x_1 + \Delta x) - f(x_1))^2)^{1/2} = \left(1 + \left(\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}\right)^2\right)^{1/2} \Delta x$$

Add this up replacing x_1 by x_i, $1 \le i \le n$ and $\Delta x = x_i - x_{i-1}$ and then let $n \to \infty$ and $\Delta x \to 0$: Arc length is approximately

$$\sum_{i=1}^{n} \left(1 + \left(\frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}\right)^2\right)^{1/2} \Delta x \to \int_a^b \left(1 + (f'(x))^2\right)^{1/2} dx$$

Define the length of the curve $y = f(x)$, $a \le x \le b$ to be

$$\int_a^b \left(1 + (f'(x))^2\right)^{1/2} dx$$

Example: Find the length of the curve $y = 2x^{3/2}$, $0 \le x \le 11$
Solution: Here $y' = 3x^{1/2}$. The length is

$$\int_0^{11} (1 + (3x^{1/2})^2)^{1/2}\, dx = \int_0^{11} (1 + 9x)^{1/2}\, dx = \frac{1}{9} \int_{x=0}^{x=11} u^{1/2}\, du$$

by substituting $u = 1 + 9x$ so that $du = 9\, dx$. The length is therefore

$$\int_0^{11} (1 + 9x)^{1/2}\, dx = \frac{1}{9}\frac{2}{3} u^{3/2}|_{x=0}^{x=11} = \frac{2}{27}(1 + 9x)^{3/2}|_0^{11} = \frac{2}{27}[100^{3/2} - 1] = 74$$

Picture.

Example: Find the length of the curve $y = \ln \sec x$, $0 \le x \le \pi/4$.
Solution: The length is $\int_0^{\pi/4} (1 + (y')^2)^{1/2}\, dx$ and $y' = \tan x$ so that the length is

$$\begin{aligned}\int_0^{\pi/4} (1 + (\tan x)^2)^{1/2}\, dx &= \int_0^{\pi/4} ((\sec x)^2)^{1/2}\, dx \\ &= \int_0^{\pi/4} \sec x\, dx = \ln|\sec x + \tan x||_0^{\pi/4} = \ln|\sqrt{2} + 1| - 0.\end{aligned}$$

Example: Find the arc length of the curve of $x = \dfrac{y^3}{6} + \dfrac{1}{2y}$ for y in the interval [1,2]?
Solution: The formula here is (interchange x and y

$$\int_1^2 (1 + \left(\frac{dx}{dy}\right)^2)^{1/2}\, dy = \int_1^2 (1 + \left(\frac{y^2}{2} - \frac{y^{-2}}{2}\right)^2)^{1/2}\, dy = \ldots = 17/12$$