6.2 Volumes by Cylindrical Shells:

Example: The region bounded by the curve $y = x^2$, $1 \le x \le 3$ and the x-axis is rotated about the y-axis. Find the volume of the solid obtained.

Solution: (Recall we rotated this same region about the x-axis and found that the solid obtained had volume $\int_1^3 \pi(x^2)^2\, dx = 242\pi/5$.) Because the region is bounded by curves $y = x^2$ and $y = 0$ that are most easily written as functions of x and yet the axis of rotation is y, it is worthwhile investigating the geometry again. A vertical line going from a point on the lower curve $(x, 0)$ to the point (x, x^2) on the upper curve sweeps out a "cylindrical shell" which is a tin can without the top and bottom. The area of this shell is $2\pi x(x^2 - 0)$, for cut the shell along a line parallel to the y-axis and flatten it out into a rectangle. If we now think of the cylindrical shell as having thickness dx then the volume is $2\pi x^3\, dx$. The volume is therefore

$$\int_1^3 2\pi x(x^2 - 0)\, dx = 2\pi \int_1^3 x^3\, dx = 2\pi \frac{1}{4} x^4|_1^3 = \frac{\pi}{2}[3^4 - 1^4] = 40\pi.$$

The same answer could be obtained by using the slicing method of the previous section but it is awkward and not recommended:

$$\int_0^1 \pi 3^2 - \pi 1^2 dy + \int_1^9 \pi 3^2 - \pi(\sqrt{y})^2\, dy$$

Example The region bounded by the curves $y = x - 1$ and $x = (y-1)^2$ is rotated about the x-axis. Find the volume of the solid obtained.

Solution: Sketch the region. The region is bounded to the right by the line and to the left by the parabola. Therefore use y as independent variable (of integration)) The curves intersect when $(y-1)^2 = y + 1$ or when $y = 0$ or $y = 3$. therefore the volume is

$$2\pi \int_0^3 y[y + 1 - (y-1)^2]\, dy = 2\pi \int_0^3 3y^2 - y^3\, dy = 2\pi[y^3 - \frac{1}{4}y^4]_0^3 = \frac{27\pi}{2}$$

Example Suppose the region in the previous example had been rotated about the y-axis. The slices would have been the appropriate method.

In general

	Region $g(x) \le y \le f(x)$	Region $g(y) \le x \le f(y)$
x-axis rotation	Slices $\pi \int_a^b (f(x))^2 - (g(x))^2\, dx$	Shells $2\pi \int_c^d y(f(y) - g(y))\, dy$
y-axis rotation	Shells: $2\pi \int_a^b x(f(x) - g(x))\, dx$	Slices $\pi \int_c^d (f(y))^2 - (g(y))^2\, dy$

The regions in the first column are sometimes referred to as "regions of Type I" and those in column 2 are "regions of Type II." We can find formulas to rotate either type of region about axes parallel to the co-ordinate axes but we do NOT consider axes of rotation that are not parallel to the coordinate axes. We can handle regions that are made up of several regions each of either Type I or II.

Example: The region enclosed by the curves $x = y^2$ and $x = y^4$. is rotated around the x-axis. Find the volume of the region obtained.

Solution: Sketch the region.

Find the intersections. $y^4 = y^2$ or $y^2(y-1)(y+1) = 0$ so that $y = -1$, $y = 0$ and $y = 1$. Since the curves are given as functions of y it is easiest to measure the horizontal distance between the curves (Type II): subtract the leftmost curve from the rightmost $y^2 - y^4$. A horizontal line of length $y^2 - y^4$ between the curves, traces out a cylindrical shell when rotated around the x-axis. The distance to the axis of rotation is y and so the area of the cylindrical shell is $2\pi y(y^2 - y^4)$. The volume of the solid is therefore

$$\int_0^1 2\pi y(y^2 - y^4)\,dy = \pi/6$$

Notice that we need only integrate $0 \leq y \leq 1$ (or alternatively $-1 \leq y \leq 0$). Each half of the region (above and below the x-axis) traces out the same solid. If we had integrated from $-1 \leq y \leq 1$ then we would have got double the correct answer.