6.1 Volumes by Cross-Sections:

In this and the next section we show how integration can be used to find the volume of certain solids. The solids must satisfy certain symmetry requirements (or we need the methods of Chapter 15). In this section we find volumes of solids withknow cross-sectional areas.

Example: A pyramid has base, a 100 meter square and height 50 meters. Find the volume.

Solution: Draw the x-axis through the axis of symmetry of the pyramid.

A plane perpendicular to the axis (and therefore parallel to the base) cuts the pyramid in a square. If the plane is x units from the apex of the pyramid then the side length of the square is $2x$ and therefore the area is $4x^2$. Therefore a slab of the pyramid cut by two parallel planes, one x units from the apex and the other $x + dx$ units from the apex (so that the slab is dx units thick) has volume approximately

$$4x^2\,dx$$

(cross-sectional area times thickness). This is only approximate because the slab is not a rectangular solid but it is accurate enough. Add up the volume of all the slabs by integrating x from 0 to 50. The volume is V where

$$V = \int_0^{50} 4x^2\,dx = \frac{4}{3}x^3|_0^{50} = \frac{4}{3}50^3 = \frac{5}{3} \times 10^5$$

so that the volume is 1.6×10^5 cubic meters.

Volumes by Cross Sections: Suppose S is a solid and that all the planes perpendicular to some axis (which we call the x-axis) have known cross sections of area: so that x units along the x-axis the area is $A(x)$ for all x, $a \leq x \leq b$. Then the volume is

$$V = \int_a^b A(x)\,dx$$

Special Case: Volumes of Rotation:

Example: The region beneath the curve $y = x^2$, $1 \leq x \leq 3$ but above the x-axis is rotated around the x-axis. Find the volume of the solid generated.

Solution: Sketch the region.

The cross sections perpendicular to the x-axis are disks of area πr^2 where the radius $r = x^2$ so that the cross-sectional area is $A(x) = \pi(x^2)^2 = \pi x^4$. Therefore the volume is

$$V = \int_1^3 \pi x^4\,dx = \pi \frac{1}{5} x^5 |_1^3 = \frac{\pi}{5}[3^5 - 1^5] = \frac{242\pi}{5}$$

Example: Find the volume of the solid obtained by rotating the region bounded by the curves

$$y = x^2 \qquad y = x + 2$$

about the x-axis.

Solution: Sketch the two curves.

Find the intersections: $x^2 = x + 2$ or $(x-2)(x+1) = 0$ so that $x = -1$ and $x = 2$ are the two points of intersection. The region between the two curves is rotated about the

x-axis. The cross section of the solid perpendicular to the x-axis is a washer or annulus with outer radius $r_{outer} = x + 2$ (the outer curve) and inner radius $r_{inner} = x^2$ (the inner curve). The area of the cross section is the difference of the area of the two circles $A(x) = \pi r_{outer}^2 - \pi r_{inner}^2 = \pi(x+2)^2 - \pi(x^2)^2$. The volume of the solid is therefore

$$\begin{aligned}\int_{-1}^{2} \pi(x+2)^2 - \pi(x^2)^2\,dx &= \pi\frac{1}{3}(x+2)^3 - \pi\frac{1}{5}x^5|-1^2 \\ &= \pi\frac{1}{3}4^3 - \pi\frac{1}{5}2^5 - (\pi\frac{1}{3}1^3 - \pi\frac{1}{5}(-1)^5 = \frac{72\pi}{5}\end{aligned}$$

Example: The region enclosed by the curves $y = x^4$ and $y = x$ is rotated about the y-axis. (a) Find the volume of the solid obtained. (b) Suppose that the axis of rotation is the line $x = -2$. Set up, but do not evaluate an integral for the volume of the solid obtained.

Solution: Sketch the curves and the region

The curves intersect when $x^4 = x$ so that $x = 0$ or $x = 1$. (a) Since the axis of rotation is the y-axis the distance to the y axis is given by x and so we solve for x: $x = y^{1/4}$ and $x = y$. The volume of the solid is

$$\int_0^1 \pi(y^{1/4})^2 - \pi y^2\,dy = \pi\int_0^1 y^{1/2} - y^2\,dy = \pi\frac{2}{3}y^{3/2} - \frac{1}{3}y^3|_0^1 = \pi(\frac{2}{3} - \frac{1}{3}) = \frac{\pi}{3}$$

(b)The new axis means that the radius of the washers is increased by 2 the volume is

$$\int_0^1 \pi(y^{1/4}+2)^2 - \pi(y+2)^2\,dy = \pi\int_0^1 y^{1/2} + 4y^{1/4} - y^2 - 4y(= 23\pi/15)$$