5.6 Area Between Curves:

Recall integration by substitution. $\int \sin(x^2)2x\,dx = ?$.

Also recall that the definite integral $\int_a^b f(x)\,dx$ of a nonnegative function $f : [a, b] \to \mathbb{R}$ is the area under the graph but above the x-axis. What is $\int_a^b f(x)\,dx$ when f is non-positive?

As an introductory application of integration we pursue the idea of finding areas. Recall that areas were a motivation for defining Riemann sums and definite integrals in §5.1 and §5.3. ($\int_a^b f(x)\,dx \approx \sum_{k=1}^n f(c_k)\Delta x_k$)

Example: (a) Find the area enclosed between the curve $y = 9 - x^2$ and the x-axis. (b) Find the area enclosed between the curve $y = 9 - x^2$ and the line $y = 5$.

Solution: Sketch the curve. It is a parabola opening down shifted up 9 units. It intersects the x-axis ($y = 0$) at $x = \pm 3$.

(a) The area between the parabola and the x-axis is

$$\int_{-3}^{3} 9 - x^2\,dx = 9x - x^3/3|_{-3}^{3} = 27 - 27/3 - (-27 + 27/3) = 36$$

(b) The area between the parabola and the horizontal line $y = 5$ is

$$\int_{-2}^{2} 9 - x^2 - 5\,dx = 4x - x^3/3|_{-2}^{2} = 8 - 8/3 - (-8 + 8/3) = 32/3$$

Formula. The area bounded by the two curves $y = f(x)$ and $y = g(x)$ between $a \le x \le b$ is

$$\int_a^b |f(x) - g(x)|\,dx$$

Example: Find the area enclosed by the two curves $y = 3x^2$ and $y = x^2 + 18$.

Solution: Sketch the two curves.

We will need the points of intersection of the two curves. Set $3x^2 = x^2+18$: $2x^2-18 = 0$ or $(x-3)(x+3) = 0$ so that $x = \pm 3$ where $y = 27$. The area is the area beneath the higher curve $y = x^2 + 18$, $-3 \leq x \leq 3$ minus the area beneath the curve $y = 3x^2$, $-3 \leq x \leq 3$:

$$\int_{-3}^{3} x^2+18-3x^2\,dx = \int_{-3}^{3} 18-2x^2\,dx = 18x - \frac{2}{3}x^3|_{-3}^{3} = 54 - \frac{2}{3}27 - (-54 - \frac{2}{3}(-27)) = 72$$

The area is 72 square units.

Example: Find the area enclosed by the two curves $y = x$ and $y = x^3$.

Solution: Sketch the two curves.

Find the intersections: $x^3 = x$ or $x(x-1)(x+1) = 0$. Three intersections $x = 0$ and $x = \pm 1$. According to our formula the area is

$$\int_{-1}^{1} |x^3 - x|\,dx$$

That can be evaluated by checking when the expression inside the $|\cdot|$ signs is zero and breaking up the integral accordingly.

$$\int_{-1}^{1} |x^3 - x|\,dx = \int_{-1}^{0} |x^3 - x|\,dx + \int_{0}^{1} |x^3 - x|\,dx = \int_{-1}^{0} x^3 - x\,dx - \int_{-1}^{0} x^3 - x\,dx$$

The area is therefore

$$\int_{-1}^{0} x^3 - x\,dx - \int_{-1}^{0} x^3 - x\,dx = x^4/4 - x^2/2|_{-1}^{0} - [x^4/4 - x^2/2]0^1 = 1/4 - [-1/4] = 1/2$$

Of course it would be possible to use symmetry but care must be taken.

Example: Find the area enclosed by the curves $x = y^2 - y$ and $y = x - 3$.

Solution: Sketch the curves.

They intersect when $y^2 - y = y + 3$ $(y-3)(y+1) = 0$ so that $y = -1$ or $y = 3$. The area is

$$\int_{-1}^{3} |y + 3 - (y^2 - y)|\, dy = \int_{-1}^{3} 2y + 3 - y^2\, dy$$

because the integrand does not change sign inside the interval of integration. Evaluate

$$\int_{-1}^{3} 2y + 3 - y^2\, dy = y^2 + 3y - y^3/3|_{-1}^{3} = 9 + 9 - 27/3 - (1 - 3 + 1/3) = 32/3$$