5.5 **Integration by Substitution:** We have learned that integration is just "antidifferentiation." Therefore to extend our ability to integrate it is appropriate to look at our differentiation rules. We begin with the chain rule. Recall

Chain Rule: $\frac{d}{dx}f(u(x)) = f'(u(x))u'(x)$

It follows that

$$\int f'(u(x))u'(x)dx = f(u(x)) + C$$

$$\int_a^b f'(u(x))u'(x)dx = f(u(x))|_a^b$$

How can we use this? We use $u(x)$ to be the "inside" funcction but we may be unlucky and not have u'

Example: Evaluate $\int \sin(x^2)2x\,dx$

Solution: Substitute $u = x^2$, $du = 2x\,dx$. (Note the differential notation.)

$$\int \sin(x^2)2x\,dx = \int \sin u\,du = -\cos u + C = -\cos x^2 + C$$

Check by differentiation: By the chain rule.

$$\frac{d}{dx} - \cos x^2 = \sin(x^2)2x$$

Observe that $\int \sin(x^2)\,dx$ cannot be evaluated by substitution or any other method.

Example: Evaluate $\int x^2\sqrt{2+x^3}\,dx$

Solution: Substitute $u = 2+x^3$ so that $du = 3x^2\,dx$ or $(1/3)du = x^2dx$ or $du/(3x^2) = dx$.

$$\begin{aligned}\int x^2\sqrt{2+x^3}\,dx &= \int x^2(2+x^3)^{1/2}\,dx \\ &= \int x^2u^{1/2}\frac{du}{3x^2} = \frac{1}{3}\int u^{1/2}du = \frac{1}{3}\frac{2}{3}u^{3/2} + C \\ &= \frac{2}{9}(2+x^3)^{3/2} + C\end{aligned}$$

Note that there must NOT be any x in the "du" integral when you actually integrate. Check by differentiation: by the chain rule.

$$\frac{d}{dx}\frac{2}{9}(2+x^3)^{3/2} = \frac{2}{9}\frac{3}{2}(2+x^3)^{1/2}3x^2$$

Example: Evaluate $\int_{-1}^{2} x^2\sqrt{2+x^3}\,dx$

Solution:

$$\int_{-1}^{2} x^2\sqrt{2+x^3}\,dx = \frac{2}{9}(2+x^3)^{3/2}|{-1}^2 = \frac{2}{9}[10^{3/2} - 1]$$

Example Evaluate $\int \frac{x^2+2}{(x^3+6x)^2}\,dx$

Solution: Substitute $u = x^3 + 6x$, so that $du = (3x^2 + 6)\,dx$

$$\int \frac{x^2+2}{(x^3+6x)^2}\,dx = \int \frac{x^2+2}{u^2}\frac{du}{3x^2+6} = \frac{1}{3}u^{-2}\,du = \frac{1}{3} - u^{-1} + C = -\frac{1}{3}\frac{1}{x^3+6x} + C$$

Check by differentiation.

Definite Integrals: Changing the limits.

Example Evaluate $\displaystyle\int_1^2 \frac{x^2+2}{(x^3+6x)^2}\,dx$

Solution: Substitute $u = x^3 + 6x$, so that $du = (3x^2 + 6)\,dx$ as before but now we also change the limits of integration: when $x = 1$, $u = 7$ and when $x = 2$, $u = 20$

$$\int_1^2 \frac{x^2+2}{(x^3+6x)^2}\,dx = \frac{1}{3}\int_7^{20} u^{-2}\,du = -\frac{1}{3}u^{-1}|_7^{20} = -\frac{1}{60} + \frac{1}{21}$$

Example: Evaluate $\displaystyle\int_0^{\pi^2/4} \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$

Solution: The inside function here is $u = x^{1/2}$ so that $du = (1/2)x^{-1/2}\,dx$ or . Also when $x = 0$, $u = 0$ and when $x = \pi^2/4$, $u = \pi/2$ so that

$$\int_0^{\pi^2/4} \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = \int_0^{\pi/2} \frac{\cos u}{x^{1/2}}\,2x^{1/2}\,du = 2\int_0^{\pi/2} \cos u\,du = 2\sin u|_0^{\pi/2} = 2$$

Alternatively one need not change the limits of integration. First find the antiderivative:

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = 2\int \cos u\,du = 2\sin u + C = 2\sin x^{1/2} + C$$

so that

$$\int_0^{\pi^2/4} \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = 2\sin x^{1/2}|_0^{\pi^2/4} = 2$$

You can partially check your answer with differentiation. Differentiate $y = 2\sin x^{1/2}$: by the chain rule the $y' = 2\cos(x^{1/2})(1/2)x^{-1/2} = \dfrac{\cos x^{1/2}}{x^{1/2}}$.

Example; Evaluate $\displaystyle\int \sec 3x \tan 3x\,dx$

Solution: Here $u = 3x$ so that $du = 3\,dx$.

$$\int \sec 3x \tan 3x\,dx = \frac{1}{3}\int \sec u \tan u\,du = \frac{1}{3}\sec u + C = \frac{1}{3}\sec 3x + C$$

Check by differentiation.