5.4 **Fundamental Theorem of Calculus Part II:** Suppose that $f(x)$ is a continuous function, $a \leq x \leq b$ and F is an antiderivative ($F'(x) = f(x)$). Then

$$\int_a^b f(x)\,dx = F(x)|_a^b = F(b) - F(a).$$

Remark: The Fundamental Theorem of Calculus relates differentiation and integration. Integrals "undo" differentiation: $\int_a^b F'(x)\,dx = F(b) - F(a)$.

Example: $\displaystyle\int_0^2 1 + 2x - x^2\,dx = [x + x^2 - \frac{1}{3}x^3]_0^2 = 2 + 2^2 - \frac{1}{3}2^3 - 0 = 10/3$

Recall the mean value theorem. If F is as in the theorem

$$\frac{F(b) - F(a)}{b - a} = F'(c)$$

or $F(b) - F(a) = F'(c)(b - a)$ for some c, $a < c < b$. By applying this theorem repeatedly with different choices of a and b we shall derive the Fundamental Theorem of Calculus.

Proof of the FTC: Choose a partition of the interval $a \leq x \leq b$. $a = x_0 < x_1 < x_2 < x_3 < \ldots < x_{n-1} < x_n = b$. Compute

$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_0) \\ &= F(x_n) - F(x_{n-1}) + F(x_{n-1}) - F(x_{n-2}) + \ldots F(x_2) - F(x_1) + F(x_1) - F(x_0) \\ &= F'(x_n^*)(x_n - x_{n-1}) + F'(x_{n-1}^*)(x_{n-1} - x_{n-2}) + \ldots + F'(x_2^*)(x_2 - x_1) + F'(x_1^*)(x_1 - x_0) \\ &= f(x_n^*)(x_n - x_{n-1}) + f(x_{n-1}^*)(x_{n-1} - x_{n-2}) + \ldots + f(x_2^*)(x_2 - x_1) + f(x_1^*)(x_1 - x_0) \end{aligned}$$

where $x_{n-1} < x_n^* < x_n$ and $x_{n-2} < x_{n-1}^* < x_{n-1}$ and so on. The right hand side is a Riemann sum. The number n of terms is arbitrary. If we let $n \to \infty$ then we have

$$F(b) - F(a) = \int_a^b f(x)\,dx$$

by a theorem in Section 4.2.

Example: Evaluate

(1) $\displaystyle\int_1^3 x^2\,dx = [\frac{1}{3}x^3]_1^3 = \frac{26}{3}$.

Picture

Remark: You don't have to worry about the constant of integration

$$\int_1^3 x^2\,dx = [\frac{1}{3}x^3 + C]_1^3 = \frac{1}{3}3^3 + C - [\frac{1}{3} + C] = 10/3$$

(2) $\displaystyle\int_{-\pi/2}^{\pi/2} \cos\theta\,d\theta = [\sin\theta]_{-\pi/2}^{\pi/2} = 2$

Picture.

(3) $\displaystyle\int_{\pi/4}^{\pi/2} \frac{1}{(\sin\theta)^2}\,d\theta = -\cot\theta|_{\pi/4}^{\pi/2} = -[0-1] = 1$
Picture.

(4) $\displaystyle\int_1^4 \sqrt{t} + \frac{1}{\sqrt{t}}\,dt = \frac{20}{3}$
(5) $\displaystyle\int_1^{\pi/4} \frac{\sin\theta}{(\cos\theta)^2}\,d\theta = \sec\theta|_0^{\pi/4} = \sqrt{2}-1$

Definite and Indefinite Integrals: The indefinite integral is the notation for general antiderivative

$$\int \frac{\sin\theta}{(\cos\theta)^2}\,d\theta = \sec\theta + C$$

whereas the definite integral is

$$\int_1^{\pi/4} \frac{\sin\theta}{(\cos\theta)^2}\,d\theta = \sec\theta|_0^{\pi/4} = \sqrt{2}-1$$

Notice that the indefinite integral depends, in some sense on θ whereas the definite integral does not.

The Fundamental Theorem of Calculus, Part 1.

Before stating the theorem, consider the following

Example: Consider the integral.

$$\begin{aligned} \int_1^3 t^2\,dt &= \frac{1}{3}t^3|_1^3 = 9 - \frac{1}{3} \\ \int_1^x t^2\,dt &= \frac{1}{3}t^3|_1^x = \frac{1}{3}x^3 - \frac{1}{3} \end{aligned}$$

Observe that the derivative of $\frac{1}{3}x^3 - \frac{1}{3}$ is x^2 which is the integrand but in a different variable.
Picture.

Theorem: Fundamental Theorem of Calculus, Part 1. Suppose that $f(x)$ is continuous on the interval $a \le x \le b$. Then

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Example: Find the derivative of

$$g(x) = \int_{-3}^{x} \sqrt{2+t+t^2}\, dt$$

Solution: $g'(x) = \sqrt{2+x+x^2}$.

Outline of Proof of FTC I: Let $g(x) = \int_a^x f(t)\, dt$. Then the difference quotient

$$\frac{g(x+h)-g(x)}{h} = \frac{1}{h}[\int_a^{x+h} f(t)\, dt - \int_a^x f(t)\, dt] - \frac{1}{h}[\int_x^{x+h} f(t)\, dt$$

is the area under the graph of $y = f(t)$, $x \leq t \leq x+h$ divided by the width. If h is very small the area is approximately $f(x)h$ for look at the picture. Let $h \to 0$

$$g'(x) = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\, dt = f(x)$$

Picture:

Remark: This says that whenever a function $f(t)$ is Riemann integrable on an interval, $a \leq t \leq b$ then it has an antiderivative ($F' = f$)

$$F(x) = \int_a^x f(t)\, dt$$

However there may or may not be a simpler expression for F.

Example: Consider $f(x) = \sin x^2$. What is an antiderivative of f?

$$g(x) = \int_2^x \sin t^2\, dt$$

but there is no simpler expression: $g(x)$ cannot be written in terms of "elementary functions."

Example: Differentiate $g(x) = \int_{-1/2}^{x} \tan\theta\, d\theta$, $-\pi/2 < x < \pi/2$

$$g'(x) = \tan x$$

Example: Differentiate $h(x) = \int_x^3 \frac{t}{3+t+t^3}\, dt$, $x > -1$.

Solution: Since $h(x) = -\int_3^x \frac{t}{3+t+t^3}\, dt$, $x > -1$. $h' = -\frac{x}{3+x+x^3}$.

Example What is wrong with the following

$$\int_{-1/2}^{1} x^{-2}\, dx = -x^{-1}|_{-1/2}^{1} = -1 - (-1/(-1/2)) = -1 - 2 = -3?$$

Notice the integrand is non negative. Replace -1/2 by 1/2 to get $\int_{-1/2}^{1} x^{-2}\, dx = 1$.

Mean Value The **mean value** of $f(x)$ on $[a, b]$ is

$$\frac{1}{b-a} \int_a^b f(x)\, dx$$

Example: The average of $f(x) = 3x + 2$, $0 \leq x \leq 2$ is

$$\frac{1}{2}\int_0^2 3x + 2\,dx = \frac{1}{2}[\frac{3}{2}x^2 + 2x]_0^2 = \frac{1}{2}[\frac{3}{2}2^2 + 4] = 5$$

Note $f(0) = 2$ and $f(2) = 8$. So 5 is really an average.

Mean Value Theorem for Integrals If $f(x)$ is continuous on $[a, b]$ then there exists c, $a < c < b$ so that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

or

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

Proof Apply the usual mean value theorem to $F(x) = \int_a^x f(t)\,dt$.

$$\frac{F(b) - F(a)}{b-a} = F'(c) = f(c)$$

Example: If $f(x) = x^2$, $1 \leq x \leq 3$ then

$$\frac{1}{3-1}\int_1^3 x^2\,dx = \frac{1}{2}\frac{1}{3}x^3|_1^3 = \frac{13}{3}$$

So there is a number $\sqrt{13/3} = c \approx 2.08$ so $1 < \sqrt{13/3} < 3$ so that $f(\sqrt{13/3}) = 13/3$.

Example: Evaluate

$$\int_{-1}^2 |x| + 2x^3\,dx$$

Solution:

$$\begin{aligned}
\int_{-1}^2 |x| + 2x^3\,dx &= \int_{-1}^0 |x|\,dx + \int_0^2 |x|\,dx + \int_{-1}^2 2x^3\,dx \\
&= \int_{-1}^0 -x\,dx + \int_0^2 x\,dx + \frac{2}{4}x^4|_{-1}^2 \\
&= -\frac{1}{2}x^2|_{-1}^0 + \frac{1}{2}x^2|_0^2 + \frac{15}{2} = \frac{1}{2} + 2 + \frac{15}{2} = 10.
\end{aligned}$$