5 Integration

We now begin the second branch of calculus, integral calculus; the first branch was differential calculus. We start by trying to solve two different physical problems that motivate integral calculus.

5.1 Areas and Distances:

1. **Area:** We want to find the area of irregularly shaped regions in the plane. To start with we shall look at the area beneath the graph of a function $f(x)$ $a \leq x \leq b$ which is only irregular on one side. Our point of departure is the notion of the area of a rectangle and that the area of two disjoint regions is the sum of the areas of the two areas.

Example: To begin with let us find the area (at least approximately) beneath the curve

$$y = 1 + 2x - x^2, \quad 0 \leq x \leq 2$$

but above the x-axis.

Solution: We approximate the area A. The region in question is contained in the box $0 \leq x \leq 2$ and $0 \leq y \leq 2$ and so it is at most 4: $A \leq 4$ It also contains the box $0 \leq x \leq 2$ and $0 \leq y \leq 1$ and so $A \geq 2$. We can do better than this by subdividing the x-interval $0 \leq x \leq 2$ into small pieces.

If we divide it into 4 pieces for example then we can approximate the area of each piece of the region $0 \leq x \leq 1/2$, $1/2 \leq x \leq 1$, $1 \leq x \leq 3/2$ and $3/2 \leq x \leq 2$. The inscribed boxes have area $1/2 \times 1 = 1/2$, $1/2 \times 7/4 = 7/8$, $1/2 \times 7/4 = 7/8$ and $1/2 \times 1 = 1/2$ so that

$$\frac{1}{2} + \frac{1}{2}\frac{7}{4} + \frac{1}{2}\frac{7}{4} + \frac{1}{2} \leq A \quad (11/4 \leq A)$$

The circumscribed boxes have total area

$$A \leq \frac{1}{2}\frac{7}{4} + \frac{1}{2}2 + \frac{1}{2}2 + \frac{1}{2}\frac{7}{4} = \frac{15}{4}$$

Next we divide the interval $0 \leq x \leq 2$ into 8 equal pieces. (There is no reason to stick to even numbers but it does reduce the number of computations we need.) Then A is larger than the sum of the areas of the 8 inscribed boxes

$$\begin{aligned} & \frac{2}{8}f(0) + \frac{2}{8}f(1/4) + \frac{2}{8}f(1/2) + \frac{2}{8}f(3/4) + \frac{2}{8}f(5/4) + \frac{2}{8}f(3/2) + \frac{2}{8}f(7/4)\frac{2}{8}f(2) \\ = & \frac{1}{4}(1 + \frac{23}{16} + \frac{7}{4} + \frac{31}{16} + \frac{31}{16} + \frac{7}{4} + \frac{23}{16} + 1) = \frac{49}{16} \end{aligned}$$

and smaller than the area of the 8 circumscribed boxes:

$$\begin{aligned} & \frac{2}{8}f(1/4) + \frac{2}{8}f(1/2) + \frac{2}{8}f(3/4) + \frac{2}{8}f(1) + \frac{2}{8}f(1) + \frac{2}{8}f(5/4) + \frac{2}{8}f(3/2) + \frac{2}{8}f(7/4) \\ = & \frac{1}{4}(\frac{23}{16} + \frac{7}{4} + \frac{31}{16} + 2 + 2 + \frac{31}{16} + \frac{7}{4} + \frac{23}{16}) = \frac{57}{16} \end{aligned}$$

Therefore

$$\frac{49}{16} \leq A \leq \frac{57}{16}$$

If we divide the interval $0 \le x \le 2$ into 16 equal pieces then the sum of the areas of the inscribed boxes is

$$\begin{aligned} & \frac{2}{16}[f(0)+f(1/8)+f(2/8)+f(3/8)+f(4/8)+f(5/8)+f(6/8)+f(7/8) \\ & +f(9/8)+f(10/8)+f(11/8)+f(12/8)+f(13/8)+f(14/8)+f(15/8)+f(16/8)] \\ = & \frac{1}{8}[1+\frac{81}{64}+\frac{23}{16}+\frac{103}{64}+\frac{7}{4}+\frac{119}{64}+\frac{31}{16}+\frac{127}{64}+\frac{127}{64}+\frac{31}{16}+\frac{119}{64}+\frac{7}{4}+\frac{103}{64}+\frac{23}{16}+\frac{81}{64}+1] \\ = & \frac{205}{64} \end{aligned}$$

whereas the total area of the circumscribed boxes is

$$\begin{aligned} & \frac{2}{16}[f(1/8)+f(2/8)+f(3/8)+f(4/8)+f(5/8)+f(6/8)+f(7/8)+f(1) \\ & +f(1)+f(9/8)+f(10/8)+f(11/8)+f(12/8)+f(13/8)+f(14/8)+f(15/8)] \\ = & \frac{1}{8}[\frac{81}{64}+\frac{23}{16}+\frac{103}{64}+\frac{7}{4}+\frac{119}{64}+\frac{31}{16}+\frac{127}{64}+2+2+\frac{127}{64}+\frac{31}{16}+\frac{119}{64}+\frac{7}{4}+\frac{103}{64}+\frac{23}{16}+\frac{81}{64}] \\ = & \frac{221}{64} \end{aligned}$$

so that

$$3.203 \approx \frac{205}{64} \le A \le \frac{221}{64} \approx 3.453$$

The correct area is $10/3 = 3.33333$.

What we are to learn is that the area between the curve $y = f(x)$ and above the x-axis and for $a \le x \le b$ can be approximated by a sum of areas of rectangles.

$$A \approx f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + \ldots + f(x_n^*)\Delta x$$

where

$$\Delta x = \frac{b-a}{n}$$

and x_i^* is any point in the ith subinterval $[x_{i-1}, x_i]$, $1 \le i \le n$ and $x_0 = a$, $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$, $\ldots x_n = a + n\Delta x = b$.

2. Distance. How far does the car go if the velocity is

time (min.)	$0 \le t \le 15$	$15 \le t \le 30$	$30 \le t \le 45$	$45 \le t \le 60$
velocity (m.p.h.)	28	46	54	60
time (min.)	$60 \le t \le 75$	$75 \le t \le 90$	$90 \le t \le 105$	$105 \le t \le 120$
velocity (m.p.h.)	66	68	64	62

Then the distance traveled is approximately

$$s \approx \frac{1}{4}28 + \frac{1}{4}46 + \frac{1}{4}54 + \frac{1}{4}60 + \frac{1}{4}66 + \frac{1}{4}68 + \frac{1}{4}64 + \frac{1}{4}62 = 112$$

Distance traveled during the time interval $a \le t \le b$

$$s \approx v(t_1^*)\Delta t + v(t_2^*)\Delta t + v(t_3^*)\Delta t + \ldots + v(t_n^*)\Delta t$$

where $\Delta = (b-a)/n$ and t_i^* is any time in the ith subinterval $[t_{i-1}, t_i]$ $1 \le i \le n$ and where $t_0 = a$, $t_1 = a + \Delta t$, $t_2 = a + 2\Delta t \ldots t_n = a + n\Delta t = b$.

5.3 Definition of the Definite Integral: Suppose the $f(x)$ is defined on an interval $a \le x \le b$. Suppose that we have a subdivision (or partition) $\mathcal{P}$ of the interval

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$$

into n subintervals, $[x_0, x_1]$, $[x_1, x_2]$, $\ldots$, $[x_{i-1}, x_i]$, $\ldots$, $[x_{n-1}, x_n]$. Define the heights of the circumscribed and inscribed boxes

$$M_i = \max\{f(x) : x_{i-1} \le x \le x_i\} \quad m_i = \min\{f(x) : x_{i-1} \le x \le x_i\}$$

Define the lower Riemann sum and upper Riemann sum

$$\begin{aligned} L(f, \mathcal{P}) &= m_1(x_1 - x_0) + m_2(x_2 - x_1) + m_3(x_3 - x_2) + \ldots + m_n(x_n - x_{n-1}) \\ U(f, \mathcal{P}) &= M_1(x_1 - x_0) + M_2(x_2 - x_1) + M_3(x_3 - x_2) + \ldots + M_n(x_n - x_{n-1}). \end{aligned}$$

(These are the total areas of the inscribed boxes ($L(f, \mathcal{P})$) and of the circumscribed ($U(f, \mathcal{P})$). Suppose that there is one and one only one number A so that

$$L(f, \mathcal{P}) \le A \le U(f, \mathcal{P})$$

for all partitions $\mathcal{P}$. Then f is said to be *Riemann integrable* and we denote

$$A = \int_a^b f(x)\,dx$$

Interpretation: (a) If $f(x) \ge 0$ then $\int_a^b f(x)\,dx$ is the area under the $y = f(x)$, $a \le x \le b$. (b) If $v(t)$ is the velocity of a particle traveling along a straight line then $\int_a^b v(t)\,dt$ is the displacement between time $t = a$ and $t = b$. (For example if $\int_a^b v(t)\,dt = 11$ then the particle moves 11 units to the right during thet time interval $a \le t \le b$.)

Theorem. If $f(x)$ is continuous $a \le x \le b$ then $f(x)$ is Riemann integrable and

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n}[f(x_1^*) + f(x_2^*) + f(x_3^*) + f(x_4^*) + \ldots + f(x_n^*)]$$

no matter what the choice of x_i^*, provided $(i-1)(b-a)/n \le x_i^* - a \le i(b-a)/n$ (that is x_i^* is in the ith subinterval.)

Example: Use Riemann sums and 16 subintervals to approximate $\int_0^2 f(x)\,dx$

$$f(x) = 1 + 2x - x^2, \quad 0 \le x \le 2$$

Solution: Recall that a general Riemann sum is

$$\int_a^b f(x)\,dx =\approx f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + \ldots f(x_{n-1}^*)\Delta x + f(x_n^*)\Delta x$$

and so, if we take the interval length to be $\Delta x = (b-a)/n$ and factor is out we have

$$\int_a^b f(x)\,dx \approx \left(\frac{b-a}{n}\right)[f(x_1^*) + f(x_2^*) + f(x_3^*) + \ldots f(x_{n-1}^*) + f(x_n^*)]$$

For this problem we take $n =?$; $a = 0$, $b = 2$, $f(x) = 1 + 2x - x^2$. We shall use the midpoint rule which is usually the more accurate rule ($x_i^* = (x_{i-1} + x_i)/2$)

$$\begin{aligned}\int_a^b f(x)\,dx &\approx \left(\frac{2}{16}\right)[f(1/16) + f(3/16) + f(5/16) + \ldots + f(31/16)] \\ &= \left(\frac{1}{8}\right)\left[\frac{287}{256} + \frac{343}{256} + \frac{391}{256} + \frac{431}{256} + \frac{463}{256} + \frac{487}{256} + \frac{503}{256} + \frac{509}{256}\right. \\ &\quad \left. + \frac{509}{256} + \frac{503}{256} + \frac{487}{256} + \frac{463}{256} + \frac{431}{256} + \frac{391}{256} + \frac{343}{256} + \frac{287}{256}\right] = \frac{3414}{1024} \\ &\approx 3.3339844\end{aligned}$$

By our earlier work we found that

$$3.203 \leq \int_0^2 1 + 2x - x^2\,dx \leq 3.453$$

The correct answer is 10/3.

Example: Evaluate the integral $\int_1^3 1 + 2x\,dx$ by interpreting it in terms of area.

Solution: Sketch $y = 1 + 2x$ This is a rectangle angle surmounted by a triangle so that the area is $6 + 4 = 10$.

Properties of the Definite Integral:

(1) $\int_a^b c\,dx = c(b - a)$
(2) $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
(3) $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
(4) $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
(5) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
(6) If $f(x) \geq 0$ the $\int_a^b f(x)\,dx \geq 0$

Physical interpretation of $\int_a^b f(x)\,dx$:

If $f(x) \geq 0$ on the entire interval $a \leq x \leq b$ then $\int_a^b f(x)\,dx$ is simply the area beneath the graph $y = f(x)$ and above the x-axis, $a \leq x \leq b$. If $f(x) \leq 0$ on the entire interval $a \leq x \leq b$ then $\int_a^b f(x)\,dx$ is simply the area beneath the graph times (-1). If $f(x)$ changes sign on the interval then the area is: (picture)

so that the regions above the axis count positive and below the axis count negative.