4.8. Antiderivatives

Example: If $f'(x) = 2x$ then what is $f(x)$? Clearly $f(x) = x^2$ works but so does $f(x) = x^2 + 11$ In fact $f(x)$ is not completely determined by its derivative and we write $f(x) = x^2 + C$ where C is a constant that is unknown. Therefore there are infinitely many solutions to our problem, one for every real number C. Are there any others?

Uniqueness: Suppose $g'(x) = 0$ for all x in an interval. What is g? By the mean value theorem if $a < b$ are any two numbers in the interval. then

$$\frac{g(b) - g(a)}{b - a} = g'(c) = 0$$

so that $g(b) = g(a)$. Since $a < b$ was arbitrary this says that $g(x)$ is constant $g(x) = C$ on teh interval.

Therefore if two functions $f(x)$ and $h(x)$ have the same derivative: $f'(x) = h'(x)$ on an interval then they differ by $f(x) - h(x) = g(x) = C$ because $g'(x) = f'(x) - h'(x) = 0$.

Example: Find the (most general) antiderivative of $f'(x) = 2x$.

Solution: $f(x) = x^2 + C$.

Example: Find the antiderivative of

1. x^3

   Solution: The antiderivative is $\frac{1}{4}x^4 + C$. Check by differentiation.

   $$\frac{d}{dx}x^4 = 4x^3$$

   divide by 4 and move the 1/4 inside the differentiation sign.

2. x^7

   Solution: The antiderivative is $\frac{1}{8}x^8 + C$. Check by differentiation.

3. $1/x^2$

   Solution: First we write $1/x^2 = x^{-2}$ The antiderivative is $-x^{-1} + C$. Check by differentiation.

General Rule: The antiderivative of x^n is

$$\frac{1}{n+1}x^{n+1} + C \quad n \neq -1$$

DANGER: We cannot yet find the antiderivative of x^{-1}.

Notation: The general antiderivative of x^n, $n \neq -1$

$$\int x^n \, dx = \frac{1}{n+1}x^{n+1} + C \quad n \neq -1$$

Examples: Find f if

1. $f'(x) = \dfrac{1}{\sqrt{x}} + 1 + \sqrt{x}$
2. $f'(x) = \sin x$
3. $f'(x) = \cos x$
4. $f'(x) = \sec^2 x$
5. $f'(x) = \tan x \sec x$
6. $f'(x) = \csc^2 x$
7. $f'(x) = \csc x \cot x$

Rules:

$$\int f(x) + g(x)\, dx = \int f(x)\, dx + \int g(x)\, dx$$

$$\int kf(x)\, dx = k \int f(x)\, dx$$

Direction Fields: An equation $f'(x) = g(x)$ assigns to each x, a values of the slope of the tangent line to the curve $y = f(x)$. That is we do not know what $y = f(x)$ is but we know how fast the curve is rising or falling.

Example: Consider $f'(x) = \cos x$. Starting at $x = 0$ say we can see how fast the curve $y = f(x)$ is rising or falling. The lines form a “direction field.” Once we choose a point on the curve we have the entire curve.

The Constant of Integration:

Example: Suppose $f'(x) = 3x^2$ and $f(1) = 3$. Then $f(x) = x^3 + C$ and $1^3 + C = f(1) = 3$. Therefore $C = 3 - 1 = 2$ and $f(x) = x^3 + 2$. That is the extra condition $f(1) = 3$ uniquely determines f.

Example: A boy throws a ball. He releases the ball with an initial velocity of 48 feet per second. Find the velocity.

Solution: We know from physics that acceleration due to gravity is 32 ft/s^2 down: $v' = -32$. Therefore $v = -32t + C$ but $48 = v(0) = -32(0) + C$ so that $C = 48$ and the velocity is

$$v(t) = -32t + 48$$