4.6 Optimization:

Example: A farmer wishes to fence a rectangular plot of land for pasture. One side of the plot will be along a neighbor's property line and will cost \$12 per yard to fence, whereas the remainder of the fence costs only \$8 per yard. She can afford at most \$2000. What is the largest area she can enclose?

Solution: Draw a picture of a rectangle with one edge along the neighbors property line. The side of the rectangle along the property line will be the shorter one. We want to maximize the area of the rectangular plot. If the plot is x yards wide and y yards long where x is the side shared with the neighbor then the area is $A = xy$. However there is a restriction on how much she can spend:

$$20x + 16y = 2000 \qquad \text{or} \qquad 5x + 4y = 500$$

Therefore

$$A = x(125 - 5x/4)$$

and $0 \leq x \leq 2000/20$. Find the absolute maximum of A. Recall the procedure from Section 4.1.

1. Critical Points. $A' = 125 - 5x/2$. Set $A' = 0$: $x = 50$. Since A' is defined everywhere there is no other critical points.
2. Endpoints. $x = 0$ $x = 100$
3. Evaluate: When $x = 0$, $A = 0$; when $x = 100$, $A = 0$. When $x = 50$, $A = 3125$ The field can be 3125 square yards.

Remark: If the farmer had not had to pay more for the side along the property line and so had just had to pay \$8 per yard for all 4 sides then the problem would have been symmetric in the length and width and so the optimal rectangle would be a square in that case.

Example: A right circular cylindrical can with no top is to hold a volume of one liter (1000cm^3). What are the dimensions of the can that requires the least amount of material?

Solution: We want to minimize material or surface area. If the height of the can is h and the radius is r is

$$S = 2\pi rh + \pi r^2$$

There is a constraint: $\pi r^2 h = 1000$.

$$S = \pi r^2 + \frac{2000}{r}$$

for $r > 0$. ($r = 0$ is not possible.) Check critical points. $S' = 2\pi r - \frac{2000}{r^2}$. Set to 0 $S' = 0$. $r^3 = 1000/\pi$ so that $r = 10(\pi)^{-1/3}$. Is this a max of min? Second derivative test. $S'' = 2\pi + 4000/r^3$ so that $S''(10(\pi)^{-1/3}) > 0$ so that $r = 10(\pi)^{-1/3}$ is a local min. But in fact it is an absolute min because the S is decreasing if $r < 10(\pi)^{-1/3}$ and decreasing thereafter.

Example: Find the distance form the point (4,1/2) to the curve $y = x^2$

Solution: We want to minimize the distance from a point (x,y) on the curve to to (4,1/2): minimize $\sqrt{(x-4)^2 + (y-1/2)^2}$. Equivalently we can minimize the square of the distance: $z = (x-4)^2 + (y-1/2)^2$. Of course $y = x^2$ so that

$$z = (x-4)^2 + (x^2 - 1/2)^2$$

There is no restriction on x. Differentiate

$$z' = 2(x-4) + 2(x^2 - 1/2)2x = 2x - 8 + 4x^3 - 2x = 4x^3 - 8$$

Set to 0. Therefore $x = 2^{1/3}$. Since $z'' = 12x^2 > 0$ this is a local min and because it is the only min, it is an absolute min.

Example: Find the volume of the largest cylinder inscribed inside a cone of height 7 and radius 3.

Solution: Sketch.

We want to maximize the volume $V = \pi r^2 h$. Eliminate all but one variable. By similar triangles

$$\frac{3}{7} = \frac{r}{7-h} \qquad \text{so that} \qquad h = 7 - \frac{7}{3}r$$

and the volume is therefore

$$V = \pi r^2 (7 - \frac{7}{3}r) = \frac{7\pi}{3}(3r^2 - r^3)$$

where $0 \le r \le 3$. Maximize V. Differentiate.

$$V' = \frac{7\pi}{3}(6r - 3r^2) = 7\pi r(2 - r)$$

so the $r = 0$ or $r = 2$. V' is defined everywhere and so $r = 0$ or $r = 2$ are the only two critical points. The endpoints are $r = 0$ and $r = 3$. Evaluate V: if $r = 0$ then $V = 0$ and if $r = 2$ then $V = 28\pi/3$ and if $r - 3$ then $V = 0$. The maximum occurs when $r = 2$ and that volume is $V = 28\pi/3$.