4.5 L'Hôpital's Rule:

L'Hôpital's Rule: Suppose $f(a) = 0$ and $g(a) = 0$ and f and g are differentiable near a. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Example: Evaluate $\lim_{x\to\pi} \frac{x-\pi}{\sin x}$

Solution: $\lim_{x\to\pi} \frac{x-\pi}{\sin x} \left(= \frac{0}{0}\right) = \lim_{x\to\pi} \frac{1}{\cos x} = -1$ Proof of L'Hôpital's Rule: Let us suppose that $g(x) \neq 0$ for $|x-a| > 0$ small enough and $g'(a) \neq 0$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} = \frac{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x\to a} \frac{g(x)-g(a)}{x-a}} = \frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Why didn't we use this rule in Chapter 1?

Example: $\lim_{x\to 3} \frac{x^2-2x-3}{x^2-9}$

Solution: $\lim_{x\to 3} \frac{x^2-2x-3}{x^2-9} \left(= \frac{0}{0}\right) = \lim_{x\to 3} \frac{2x-2}{2x} = \frac{4}{6}$

Example: $\lim_{x\to 0} \frac{1-\cos x}{x\sin x}$

Solution:

$$\begin{aligned}
\lim_{x\to 0} \frac{1-\cos x}{x\sin x} \left(= \frac{0}{0}\right) &= \frac{\sin x}{x\cos x + \sin x} \left(= \frac{0}{0}\right) \\
&= \frac{\cos x}{-x\sin x + \cos x + \cos x} = \frac{1}{2}.
\end{aligned}$$

Example $lim_{x\to 0} \cot x - \frac{1}{x}$

Solution: This is of the form $\pm\infty - \pm\infty$ which means there may be enough cancellation that anything can happen. Find a common denominator.

$$\begin{aligned}
\lim_{x\to 0} \cot x - \frac{1}{x} &= \lim_{x\to 0} \frac{x\cos x - \sin x}{x\sin x} \left(= \frac{0}{0}\right) \\
&= \lim_{x\to 0} \frac{\cos x - x\sin x - \cos x}{\sin x + x\cos x} \left(= \frac{0}{0}\right) \\
&= \lim_{x\to 0} \frac{-\sin x - x\cos x}{\cos x + \cos x - x\sin x} = 0
\end{aligned}$$

Example: $\lim_{x\to\infty} \frac{x}{1-e^x}$

Solution: This expression is of the form $\pm\infty/\infty$. Again there is a version of l'Hospital's rule that applies here

$$\lim_{x\to\infty} \frac{x}{3-e^x} \left(= \pm\frac{\infty}{\infty}\right) = \lim_{x\to\infty} \frac{1}{-e^x} = 0.$$

L'Hôpital's Rule

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

applies if

1. $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$
2. $\lim_{x\to a} f(x) = \pm\infty = \lim_{x\to a} g(x)$