4.3 Graphing:

Increasing/Decreasing Test: Suppose

1. $f'(x) > 0$, $a \le x \le b$. Then $f(x)$ is increasing $a \le x \le b$.
2. $f'(x) < 0$, $a \le x \le b$. Then $f(x)$ is decreasing $a \le x \le b$.

Verification Suppose $f'(x) > 0$, $a \le x \le b$. Suppose $a \le \tilde{a} < \tilde{b} \le b$. Then by the Mean Value Theorem there is c so that

$$\frac{f(\tilde{b}) - f(\tilde{a})}{\tilde{b} - \tilde{a}} = f'(c) > 0$$

so $f(b) > f(a)$.

Example: Find the intervals on which $f(x) = x^3 + 3x^2 - 9x - 12$ is increasing and decreasing. Sketch the graph of f

Solution: Differentiate $f'(x) = 3x^2 + 6x - 9$. We must decide where the derivative is positive and negative: check for critical points. Transition from increasing to decreasing or back can only occur at a critical point. Check $f'(x) = 0$

$$3x^2 + 6x - 9 = 0 \qquad \text{or} \qquad x^2 + 2x - 3 = 0$$

Factor $(x + 3)(x - 1) = 0$. So $x = 1$ and $x = -3$ are critical points. They are the only critical points. They divide the domain of $f(x)$ into intervals:

Interval	Evaluate f'	Increasing or Decreasing
$-\infty < x < -3$	$f'(-4) = 15$	Increasing
$-3 < x < 1$	$f'(0) = -3$	Decreasing
$1 < x < \infty$	$f'(2) = 15$	Increasing

Now sketch the graph. We have already discovered that $x = -3$ is an important point $f(-3) = 15$ and similarly $f(1) = -16$. In fact $x = -3$ is a local max and $x = 1$ is a local min.

First Derivative Test: Suppose that c is a critical point ($f'(c) = 0$ or $f'(c)$ DNE). Then

1. If f' changes from positive to negative at $x = c$ then f has a local max at $x = c$.
2. If f' changes from negative to positive at $x = c$ then f has a local min at $x = c$.
3. If f' does not change sign at c then f does NOT have a local max nor min at $x = c$.

Example: Find the local max and min of $f(x) = x^3 + 3x^2 - 9x - 12$ of the previous example. Check the critical points $x = -3$ and $x = 1$. We saw that for $x < -3$, $f'(x) > 0$ because $f'(-4) > 0$ and we saw that for $-3 < x < 1$, $f'(x) < 0$ because $f'(0) < 0$. Therefore $x = -3$ is a local minimum. Note we dervie this without needing the graph. Similarly for the critical point $x = 1$. We just saw that $f'(x) < 0$ for $x < 1$. Since $f'(2) > 0$, f is increasing for $x > 1$ and so $x = 1$ is a local minimum point.

Example: $f(x) = x^3$ has a critical point at $x = 0$ but it is neither a local max nor min because $f'(x) \geq 0$.

Concavity: We say $f(x)$ is *concave up* if the slope of the tangent line is increasing and *concave down* if the slope of the tangent line is decreasing.

Picture

Concavity Test:

1. If $f''(x) > 0$ then f is concave up.
2. If $f''(x) < 0$ then f is concave down.

Definition: Inflection Point. A point $(c, f(c))$ on a graph is an inflection point if $f''(c) = 0$.

Example: For the function $f(x) = x^4 - 6x^2$

1. Find the intervals of increase and decrease.
2. Find the intervals of concavity and points of inflection.
3. Find the local max and min values.
4. Sketch the graph of $f(x)$.

Solution: Differentiate. $f'(x) = 4x^3 - 12x$ and we also need $f''(x) = 12x^2 - 12$.

1. Check for critical points. $f'(x) = 0$: $x = 0$ and $x = \pm\sqrt{3}$. $f'(x)$ is defined everywhere so there are just the 3 critical points.

Interval	Evaluate f'	Increasing or Decreasing
$-\infty < x < -\sqrt{3}$	$f'(-2) = -8$	Decreasing
$-\sqrt{3} < x < 0$	$f'(-1) = 8$	Increasing
$0 < x < \sqrt{3}$	$f'(1) = -8$	Decreasing
$\sqrt{3} < x < \infty$	$f'(2) - 8$	Increasing

2. Check for inflection points: $f''(x) = 0$. $x = \pm 1$. Check also where $f''(x)$ is not defined. There are no such points.

Interval	Evaluate f''	Concavity
$-\infty < x < -1$	$f''(-2) = 36$	Up
$-1 < x < 1$	$f''(0) = -12$	Down
$1 < x < \infty$	$f''(2) = 36$	Up

3. The local extrema occur at critical points: $x = -\sqrt{3}$ is local min; $x = 0$ is a local max; and $x = \sqrt{3}$ is a local min.

4. Sketch. Mark the critical points and points of inflection on the x-axis and then the intervals of increase and decrease and of cancave up and concave down. Find also the points that the curve passes through at the critical points and points of inflection: $(-\sqrt{3}, -9)$, $(\sqrt{3}, -9)$, $(0, 0)$, $(-1, -5)$, $(1, -5)$.

Second Derivative Test for Max/Min: Suppose $f''(x)$ is continuous near $x = c$ and $f'(c) = 0$

1. If $f''(c) > 0$ then c is a local min

2. If $f''(c) < 0$ then c is a local max.

Example: Find the critical points of $f(x) = (2 - x)\sqrt{x}$ and use both the first and second derivative tests to determine whether or not these are local max, min or neither.

Solution Notice $f(x) = 2x^{1/2} - x^{3/2}$ is defined only for $x \geq 0$. Differentiate:

$$f'(x) = x^{-1/2} - (3/2)x^{1/2} = \frac{2 - 3x}{x^{1/2}}$$

. Critical points: Set $f'(x) = 0$ at $x = 2/3$. Also $f'(x)$ does not exist at $x = 0$.

Second Derivative Test: $f''(x) = -(1/2)(2-x)^{-1/2} - (1/2)(2-x)^{-1/2} + (1/4)x(2-x)^{-3/2}(-1)$ or $f''(x) = -(2-x)^{-1/2} - (1/4)x(2-x)^{-3/2}$ and $f''(4/3) < 0$ so that $x = 4/3$ is a local max. But $f''(2)$ does not exist.

First Derivative Test. $f'(1) = 1 - 1/2 > 0$ so that f is increasing for $-\infty < x < 4/3$. $f'(3/2) = (1/2)^{1/2} - (1/2)(3/2)2^{1/2} < 0$ so that f is decreasing $4/3 < x < 2$ so that $x = 2$ is an endpoint minimum.

Asymptotes: An asymptote to a curve is a straight line that gets arbitrarily close to the curve far away from the origin.

Vertical Asymptotes: Recall that a vertical straight line is given by $x = c$, for example $x = -4$ or $x = 7$.

Example: Find all the vertical asymptotes of the curve

$$y = \frac{x}{x-2}$$

Solution: Look for division by 0: x=2.

Example: Find all the vertical asymptotes of the curve

$$y = \frac{2}{x^3+1}$$

Solution: Factor the bottom $x^3+1 = (x+1)(x^2-x+1)$. The quadratic is irreducible and so $x = 2$ is the only asyptote.

Horizontal Asymptotes and Limits at Infinity. Recall that a horizontal line has an equation y = constant. Horizontal asymptotes are related to limits at infinity: for example if

$$\lim_{x\to\infty} f(x) = 4$$

then $y = 4$ is a horizontal asymptote (at ∞).

Examples:

1. $\lim\limits_{x\to\pm\infty} 1/x = 0$ so that $y = 1/x$ has horizontal asymptote $y = 0$.

2. $\lim_{x\to\pm\infty} \frac{x}{x-1} = \lim_{x\to\pm\infty} \frac{x}{x}\frac{1}{1-1/x} = 1$

3. $\lim_{x\to\pm\infty} \frac{2x^3-4x^2}{5x^3+6x+11} = \lim_{x\to\pm\infty} \frac{x^3}{x^3}\frac{2x-4/x}{5+6/x^2+1/x^3} = \frac{2}{5}.$

4. $\lim_{x\to\pm\infty} \frac{2x^3-4x^2}{5x^2+11} = \lim_{x\to\pm\infty} \frac{x^2}{x^2}\frac{2x-4}{5+11/x^2} = \infty$ There is no horizontal asymptote.

5. $\lim_{x\to\pm\infty} \frac{5x^2+11}{2x^3-4x^2} = \lim_{x\to\pm\infty} \frac{x^2}{x^2}\frac{5+11/x^2}{2x-4} = 0$ so that $y = 0$ is a horizontal asymptote.

Examples: Sketch the graph of $f(x) = \frac{x^2+x-2}{x^2}$. In addition, find

1. the asymptotes
2. the intervals of increase and decrease.
3. the intervals of concavity and points of inflection.
4. the local max and min values.

Solution: Vertical asymptote at $x = 0$; horizontal asymptote at $y = 1$. This means that $x = 0$ is an endpoint of the domain. The function can change from increasing to decreasing (of the reverse) at $x = 0$ and similarly concavity can change. Compute the derivatives. First simplify

$$f(x) = 1 + \frac{1}{x} - \frac{2}{x^2} = 1 + x^{-1} - 2x^{-2}$$

so that

$$f'(x) = -x^{-2} + 4x^{-3} = \frac{4-x}{x^3} \qquad f''(x) = 2x^{-3} - 12x^{-4} = \frac{2x-12}{x^4}$$

Intervals of Increase and Decrease: Critical points $x = 4$. ($f'(x)$ is defined everywhere but $x = 0$ but this is not a critical point.)

Interval	Evaluate f'	Increasing or Decreasing
$-\infty < x < 0$	$f'(-1) = -5$	Decreasing
$0 < x < 4$	$f'(1) = 3$	Increasing
$4 < x < \infty$	$f'(5) = -1/125$	Decreasing

Intervals of Concavity: Inflection Points $f''(x) = 0$ $x = 6$. ($f''(x)$ is defined everywhere.)

Interval	Evaluate f''	Concavity
$-\infty < x < 0$	$f''(-1) = -14$	Down
$0 < x < 6$	$f''(1) = -10$	Down
$6 < x < \infty$	$f''(7) > 0$	Up

Local Max/Min: At $x = 4$ there is a local max. Sketch the graph