**4.1 Maxima and Minima:**

**Definition: Local Maximum and Minimum** A function $f(x)$ has a *local maximum* at $x = c$ if there is an open interval $I$ containing $c$ so that $f(x) \leq f(c)$ for all $x$ in $I$. Similarly $f(x)$ has a *local minimum* at $x = c$ if there is an open interval $I$ containing $c$ so that $f(x) \geq f(c)$ for all $x$ in $I$.

Picture:

Observe that $f(x)$ can have a local max (or min) at an endpoint $a$ say ("endpoint max") of its domain $D$ provided $a$ is in $D$: just replace "all $x$ in $I$" above definition by "all $x$ in $I$ and $D$"

**Fermat's Theorem:** If $c$ is a local maximum or minimum for $f(x)$ but is not an endpoint of the domain of $f$ and if $f'(c)$ exists then

$$f'(c) = 0$$

**Verification:** Suppose to be specific that $c$ is a local min. Then $f(c+h) - f(c) \geq 0$ if $h > 0$ and so

$$\frac{f(c+h) - f(c)}{h} \geq 0 \qquad \text{if } h > 0$$

Therefore $f'(c) = \lim_{h\to 0+}(f(c+h) - f(c))/h \geq 0$ But by the same reasoning

$$\frac{f(c+h) - f(c)}{h} \leq 0 \qquad \text{if } h < 0$$

Therefore $f'(c) = \lim_{h\to 0-}(f(c+h) - f(c))/h \leq 0$. This leaves $f'(c) = 0$ as the only possibility.

**Example:** Consider the function $f(x) = 2x - x^2$. Since $f$ is differentiable everywhere, the local maxima and minima if they occur must occur where $f'(x) = 0$. Here $f'(x) = 2 - 2x$ so that $f'(x) = 0$ at $x = 1$ and nowhere else. Therefore there is at most one local max or min at $x = 1$. (Completing the square $f(x) = -(x-1)^2 + 1$ we see it is a local max.)

**Definition:** $f(x)$ is said to have a *critical* point at $x = c$ if either $f'(c) = 0$ or $f(x)$ is not differentiable at $x = c$.

**Example:** Find the critical points for the function

$$y = x^2 + \frac{2}{x}$$

**Solution:** Differentiate $y' = 2x - 2x^{-2}$. The only critical point is $x = 1$.

**Definition;** (Absolute Max and Min) A function $f(x)$ is said to have an absolute maximum (resp. minimum) at $c$ if $f(x) \leq f(c)$ (resp. $f(x) \geq f(c)$) for all $x$ in the domain of $f$. $c$ is the absolute maximum *point* and $f(c)$ is the absolute maximum *value.*

**Theorem:** (Extreme Values) If $f(x)$ is continuous on a closed bounded interval $[a, b]$ then there exist points $c$ and $d$ in $[a, b]$ and $c$ is an absolute maximum point and $d$ is an absolute minimum.

**Counterexample:** $f(x) = x$, $1 < x < 2$ has no absolute max or min.

**Counterexample:** $f(x) = 1/x$, $1 \leq x < \infty$ has an absolute max at $c = 1$ but no absolute min

**Counterexample:** $f(x) = 1/x$, $-1 \leq x \leq 1$ has no absolute max nor min. Graph.

**Example:** Find the absolute maximum and minimum values of $f(x) = 2 + 3x - x^3$, $-2 \leq x \leq 3$

**Solution:** We will need the derivative $f'(x) = 3 - 3x^2$

**Closed Interval Method:**

1. Find the critical points. Solve $f'(x) = 0$: $3 - 3x^2 = 0$ or $x = \pm 1$. Also check where $f'(x)$ does not exist. No such points
2. Check the end points $x = -2$ and $x = 3$.
3. Compare the values of $f$ at the points found above. $f(-2) = 2 + 3(-2) - (-2)^3 = 4$; $f(-1) = 0$ and $f(1) = 4$ and $f(3) = -16$.

Therefore the absolute maximum value is 4 occurs at both $x = -2$ and $x = 1$ and the absolute minimum point is $x = 3$ and the absolute minimum value is -16.

**Example:** Find the absolute maximum and minimum of $f(x) = x^{2/3}$ $-1 \leq x \leq 8$.

**Solution:** We will need the derivative

$$f'(x) = (2/3)x^{-1/3} = \frac{2}{3x^{1/3}}$$

**Closed Interval Method:**

1. Find the critical points. $f'(x) = 0$:

   $$\frac{2}{3x^{1/3}} = 0$$

   No solution. Critical points ....
2. Endpoints $x = -1$ and $x = 8$
3. Evaluate. $f(-1) = (-1)^{2/3} = 1$. $f(8) = 4$. But $f(0) = 0$. 0 is a critical point.

Therefore $f$ takes the absolute maximum value 4 at the point $x = 8$. The absolute minimum value is 0 at the point $x = 0$.

**Picture:**