3.7 Implicit Differentiation:

Until now we have considered only functions $f(x)$ which are defined explicitly but now we consider functions that are defined implicitly. For example

$$x^2 + y^2 = 169$$

is the equation of a circle of radius 13.

From the picture of the it is clear that the circle is not the graph of a function but the circle can be broken into pieces which are the graph of a function and those pieces have tangent lines. Indeed $y = \pm(36 - x^2)^{1/2}$ in this case. Other examples of functions defined implicitly are $x^3 + y^3 = 6xy$ which is discussed in the text and $x^4 + y^4 - 2x^3 - 2xy^2 = x^2 + y^2$. In these examples it is not reasonable to expect to find y explicitly by a formula but the curve if we plot it still looks like it should have a tangent line.

Example Find the slope of the tangent line to the curve

$$x^2 + y^2 = 169.$$

Additionally find an equation for the tangent line to the curve at (12,-5)

Solution: Regard $y(x)$ as a function of x and differentiate "implicitly" in the x variable.

$$2x + 2y\frac{dy}{dx} = 0$$

by the generalized power rule. The slope is therefore

$$\frac{dy}{dx} = -\frac{x}{y}$$

at any point (x,y) on the curve. In particular at (12,-5) the slope is 12/5 and an equation for the tangent line is

$$y - (-5) = \frac{12}{5}(x - 12) \qquad \text{or} \qquad y = \frac{12}{5}x - \frac{169}{5}$$

Solution 2: (Not So Good) $y = \pm(169 - x^2)^{1/2}$. For the "+" case (upper half of circle)

$$\frac{dy}{dx} = \frac{1}{2}(169 - x^2)^{-1/2}(-2x) = -x(169 - x^2)^{-1/2} = -\frac{x}{y}$$

and similarly for the "$-$" case

$$\frac{dy}{dx} = -\frac{1}{2}(169 - x^2)^{-1/2}(-2x) = x(169 - x^2)^{-1/2} = -\frac{x}{y}$$

Example: Find dy/dx if

$$x^3 + x^2y + xy^2 + y^3 = 4$$

Solution Regard $y = y(x)$ and differentiate implicitly in x.

$$3x^2 + (x^2\frac{dy}{dx} + 2xy) + (x\frac{d}{dx}y^2 + y^2) + \frac{d}{dx}y^3 = 0$$

by the product rule. By the generalized power rule we have

$$3x^2 + x^2\frac{dy}{dx} + 2xy + x(2y\frac{dy}{dx}) + y^2 + 3y^2\frac{dy}{dx} = 0$$

and now we solve for the slope dy/dx in terms of x and y.

$$\begin{aligned} (x^2 + 2xy + 3y^2)\frac{dy}{dx} &= -3x^2 - 2xy - y^2 \\ \frac{dy}{dx} &= \frac{-3x^2 - 2xy - y^2}{x^2 + 2xy + 3y^2} \end{aligned}$$

Example: Find dy/dx if

$$\frac{1}{x^2} + \frac{1}{y^2} = xy$$

Solution Regard $y = y(x)$ and differentiate implicitly in x.

$$-2x^{-3} - 2y^{-3}\frac{dy}{dx} = x\frac{dy}{dx} + y$$

so that

$$\begin{aligned} (-2y^{-3} - x)\frac{dy}{dx} &= 2x^{-3} + y \\ \frac{dy}{dx} &= -\frac{2x^{-3} + y}{2y^{-3} + x} \end{aligned}$$

Fractional Powers: Example: Differentiate $y = x^{2/3} = (x^2)^{1/3} = (x^{1/3})^2$. We already decided that we could just use the power rule for fractional powers but now we can actually see why. Implicitly differentiate in the equation

$$y^3 = x^2$$

to get

$$3y^2\frac{dy}{dx} = 2x \quad \text{or} \quad \frac{dy}{dx} = \frac{2x}{3y^2} = \frac{2x}{3(x^{2/3})^2} = \frac{2}{3}x^{1-4/3}$$

just as the power rule predicts. This justifies the use of the power rule that we have assumed since Section 3.3.