3.6 Chain Rule:

Differentiation of Composed Functions

Examples of Composed Functions. Express the functions below in the from $f \circ g$, that is find f and g.

1. $\sqrt{2x+1}$

2. $\cos x^2$

3. $(\sin x)^3$

Chain Rule: Suppose that $g(x)$ is differentiable at $x = a$ and $f(x)$ is differentiable at $b = g(a)$. The $f \circ g(x)$ is differentiable at $x = a$ and its derivative is

$$(f \circ g)'(a) = f'(g(a))g'(a)$$

or if we denote $y = f(x)$ and $u = g(x)$ then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

We will verify the rule in a moment but let's first apply it. Differentiate

1. $y = \sqrt{2x+1}$

2. $h(x) = \cos x^2$

3. $g(x) = (\sin x)^3$

Solution: (1) As usual we write $y = (2x+1)^{1/2}$. Here $f(x) = x^{1/2}$ and $g(x) = 2x+1$. Derivative of f is $f'(x) = (1/2)x^{-1/2}$ so that

$$\frac{dy}{dx} = \frac{1}{2}(2x+1)^{-1/2}2$$

(because $g' = 2$.)

(2) $h'(x) = -\sin(x^2)2x = -2x\sin(x^2)$

(3) $g'(x) = 3(\sin x)^2 \cos x$

Verification of the Chain Rule: We know that

$$g'(a) = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$$

or equivalently if we define

$$E_g(h) = (g(a+h) - g(a))/h - g'(a)$$

then $\lim_{h \to 0} E_g(h) = 0$. Solve the equation for $E_g(h)$

$$g(a+h) - g(a) = g'(a)h + hE_g(h)$$

Similar reasoning applies to f: define

$$E_f(k) = (f(b+k) - f(b))/k - f'(b).$$

Then $\lim_{k\to 0} E_f(k) = 0$ and

$$f(b+k) - f(b) = f'(b)k + kE_f(k)$$

We want to find the limit as $h \to 0$

$$\frac{f(g(a+h)) - f(g(a))}{h}$$

Let $b = g(a)$ and $k = g(a+h) - g(a)$ so that $g(a+h) = b+k$. Observe also, by the continuity of g that k converges to 0 as h does. We have

$$\begin{aligned}\frac{f(g(a+h)) - f(g(a))}{h} = \frac{1}{h}(f(b+k) - f(b)) &= \frac{1}{h}(f'(b)k + kE_f(k)) \\ &= \frac{g(a+h) - g(a)}{h}(f'(b) + E_f(k)) \\ &= [g'(a) + E_g(h)](f'(b) + E_f(k))\end{aligned}$$

Now let $h \to 0$ which implies $k \to 0$

$$\lim_{h\to 0} \frac{f(g(a+h)) - f(g(a))}{h} = \lim_{h\to 0}[g'(a)+E_g(h)](f'(b)+E_f(k)) = g'(a)f'(b) = f'(g(a))g'(a)$$

which completes the proof.

Examples: Differentiate

1. $\tan 5x$
2. $y = (x^2 + 2x)^{5/4}$
3. $y = u^{-1/2}$, $u = x^2 + 11$

Solution:

1.

$$\frac{d}{dx} \tan 5x = (\sec 5x)^2 5 = 5(\sec 5x)^2.$$

2.

$$\frac{dy}{dx} = (5/4)(x^2 + 2x)^{1/4}(2x + 2) = (5/2)(x + 1)(x^2 + 2x)^{1/4}.$$

3.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -\frac{1}{2}u^{-3/2}2x = -x(x^2 + 11)^{-3/2}$$

Generalized Power Rule: $\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$
Example: Differentiate

1. $y = (\sec x)^5$
2. $y = \csc(2x+1)$

Solution:
1.

$$\frac{d}{dx}(\sec x)^5 = 5(\sec x)^4 \sec x \tan x$$

2.

$$\frac{d}{dx}\csc(2x+1) = -\csc(2x+1)\cot(2x+1)2 = -2\csc(2x+1)\cot(2x+1).$$