3.5 Differentiation Formulas for Trig Functions: Sine and Cosine: Recall from Section 2.4 that

$$\lim_{\theta \to 0} \frac{\sin h}{h} = 1$$

(by the Squeeze Theorem) and that we derived

$$\begin{aligned}
\lim_{h \to 0} \frac{\cos h - 1}{h} &= \lim_{h \to 0} \frac{(\cos h - 1))(\cos h + 1)}{h(\cos h + 1)} \\
&= \lim_{h \to 0} \frac{(\cos h)^2 - 1}{h(\cos h + 1)} \\
&= \lim_{h \to 0} \frac{-(\sin h)^2}{h(\cos h + 1)} = \lim_{h \to 0} \frac{-\sin h}{h} \sin h \frac{1}{\cos h + 1} = -1(0)(1/2) = 0
\end{aligned}$$

Recall also the sum of the angles formula for $\sin x$ and $\cos x$:

$$\begin{aligned}
\sin(A + B) &= \sin A \cos B + \sin B \cos A \\
\cos(A + B) &= \cos A \cos B - \sin A \sin B
\end{aligned}$$

Compute the difference quotient

$$\begin{aligned}
\frac{\sin(x + h) - \sin x}{h} &= \frac{\sin x \cos h + \sin h \cos x - \sin x}{h} \\
&= \frac{\sin x(\cos h - 1)}{h} + \frac{\sin h \cos x}{h} = \sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h}
\end{aligned}$$

Now let h approach 0.

$$\frac{d}{dx} \sin x = \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h} = \sin x \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \to 0} \frac{\sin h}{h} = \cos x$$

so that

$$\frac{d}{dx} \sin x = \cos x$$

Similarly we can compute the difference quotient.

$$\begin{aligned}
\frac{\cos(x + h) - \cos x}{h} &= \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\
&= \frac{\cos x(\cos h - 1)}{h} - \frac{\sin h \sin x}{h} = \cos x \frac{\cos h - 1}{h} - \sin x \frac{\sin h}{h}
\end{aligned}$$

Now let h approach 0.

$$\frac{d}{dx} \cos x = \lim_{h \to 0} \frac{\cos(x + h) - \cos x}{h} = \cos x \lim_{h \to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h \to 0} \frac{\sin h}{h} = -\sin x$$

so that

$$\frac{d}{dx} \cos x = -\sin x$$

Example: Find the derivative of $\tan x$.

Solution: Recall that $\tan x = \dfrac{\sin x}{\cos x}$ so that by the quotient rule

$$\begin{aligned}\frac{d}{dx}\tan x = \frac{\cos x \frac{d}{dx}\sin x - \sin x(\frac{d}{dx}\cos x)}{(\cos x)^2} &= \frac{\cos x \cos x - \sin x(-\sin x)}{(\cos x)^2} \\ &= \frac{1}{(\cos x)^2} = (\sec x)^2\end{aligned}$$

Example: Find the derivative of $\cot x$.

Solution: Recall that $\cot x = \dfrac{\cos x}{\sin x}$ so that by the quotient rule

$$\begin{aligned}\frac{d}{dx}\cot x = \frac{\sin x \frac{d}{dx}\cos x - \cos x(\frac{d}{dx}\sin x)}{(\sin x)^2} &= \frac{\sin x(-\sin x) - \cos x(\cos x)}{(\sin x)^2} \\ &= \frac{-1}{(\sin x)^2} = -(\csc x)^2\end{aligned}$$

Example: Find the derivative of $\sec x = 1/\cos x$

Solution: By the reciprocal rule

$$\frac{d}{dx}\sec x = \frac{-\frac{d}{dx}\cos x}{(\cos x)^2} = \frac{\sin x}{(\cos x)^2} = \sec x \tan x$$

Example: Find the derivative of $\csc x = 1/\sin x$

Solution: By the reciprocal rule

$$\frac{d}{dx}\csc x = \frac{-\frac{d}{dx}\sin x}{(\sin x)^2} = \frac{-\cos x}{(\sin x)^2} = -\csc x \cot x$$

Function	Derivative
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$(\sec x)^2$
$\cot x$	$-(\csc x)^2$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$