3.3 Product and Quotient Rules: How do you differentiate a product of two functions such as $x \sin x$? We know the derivative of $f(x) = x$ and of $g(x) = \sin x$. What about the derivative of $f(x)g(x) = x \sin x$?

PRODUCT RULE: Assume that both f and g are differentiable.

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

Example:

$$\frac{d}{dx}[x \sin x] = x\frac{d}{dx}[\sin x] + \sin x \frac{d}{dx}[x] = x \cos x + \sin x$$

Proof of the Product Rule. Recall that a differentiable function f is continuous because

$$\lim_{x\to a} f(x) - f(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}(x - a) = \left[\lim_{x\to a} \frac{f(x) - f(a)}{x - a}\right][\lim_{x\to a} x - a] = f'(a)0 = 0$$

so that $\lim_{h\to 0} f(a + h) = f(a)$.

Compute now the difference quotient for the product.

$$\begin{aligned}
&\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)[g(x+h) - g(x)] + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} f(x+h)\frac{g(x+h) - g(x)}{h} + \frac{f(x+h) - f(x)}{h}g(x) \\
&= \lim_{h\to 0} f(x+h) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}g(x) \\
&= f(x)[\frac{d}{dx}g(x)] + g(x)[\frac{d}{dx}f(x)]
\end{aligned}$$

Example:

$$\begin{aligned}
&\frac{d}{dx}[(x^4 + 5x^2 + 5)(x^3 + x^{-3})] \\
&= (x^4 + 5x^2 + 5)\frac{d}{dx}(x^3 + x^{-3}) + (x^3 + x^{-3})\frac{d}{dx}[x^4 + 5x^2 + 5] \\
&= (x^4 + 5x^2 + 5)(3x^2 - 3x^{-4}) + (x^3 + x^{-3})(4x^3 + 10x)
\end{aligned}$$

Example: $\frac{d}{dx}x^{3/2} \cos x = x^{3/2}(-\sin x) + \cos x[(3/2)x^{1/2}] = x^{1/2}[(3/2) \cos x - x \sin x]$

QUOTIENT RULE: Assume that both f and g are differentiable.

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Example:

$$\begin{aligned}\frac{d}{dx}\frac{x^3+4x}{x^2+5} &= \frac{(x^2+5)\frac{d}{dx}[x^3+4x]-(x^3+4x)\frac{d}{dx}[x^2+5]}{[x^2+2]^2}\\ &= \frac{(x^2+5)[3x^2+4]-(x^3+4x)[2x]}{[x^2+5]^2}\end{aligned}$$

Reciprocal Rule A special case of the quotient rule where $f(x)=1$.

$$\frac{d}{dx}\frac{1}{g(x)} = \frac{-\frac{d}{dx}g(x)}{[g(x)]^2}$$

Example:

$$\frac{d}{dx}\frac{1}{x^4} = \frac{-4x^3}{[x^4]^2} = \frac{-4}{x^5} = -4x^{-5}$$

This is of course the power rule for a negative exponent. We will assume the power rule works for all exponents.

Proof of the Reciprocal Rule: Compute

$$\begin{aligned}\lim_{h\to 0}\frac{\frac{1}{g(x+h)}-\frac{1}{g(x)}}{h} &= \lim_{h\to 0}\frac{\frac{g(x)}{g(x)g(x+h)}-\frac{g(x+h)}{g(x)g(x+h)}}{h}\\ &= \lim_{h\to 0}\frac{g(x)-g(x+h)}{hg(x)g(x+h)}\\ &= -\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\lim_{h\to 0}\frac{1}{g(x)g(x+h)}\\ &= \frac{-\frac{d}{dx}g(x)}{[g(x)]^2}\end{aligned}$$

Proof of the Quotient Rule: Apply the product and reciprocal rule to

$$\frac{f(x)}{g(x)} = f(x)\frac{1}{g(x)}$$

Example: Differentiate $(13\sqrt{x}+2)/(x^2-5x)$

$$\frac{d}{dx}\frac{13x^{1/2}+2}{x^2-5x} = \frac{(x^2-5x)\frac{13}{2}x^{-1/2}-(13x^{1/2}+2)(2x-5)}{[x^2-5x]^2}$$

Example Differentiate $y = \dfrac{e^x}{e^x+x}$

Solution: By the quotient rule

$$y' = \frac{(e^x+x)e^x-e^x(e^x+1)}{(e^x+x)^2} = \frac{e^x(x-1)}{(e^x+x)^2}$$