3.3 Differentiation Formulas:

We want to speed up the computation of the derivative

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

in the cases that arise the most frequently.

Example: Find the derivative of $f(x) = x^3$.

Solution: Compute $(f(x+h) - f(x))/h$. Since $f(x) = x^3$

$$f(x+h) = (x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$$

so that

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\
&= \frac{3x^2h + 3xh^2 + h^3}{h} \\
&= \frac{h(3x^2 + 3xh + h^2}{h} = 3x^2 + 3xh + h^2
\end{aligned}$$

Therefore the derivative is

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} 3x^2 + 3xh + h^2 = 3x^2$$

Example: Find the derivative of $f(x) = x^n$, $n = 0, 1, 2, 3, \ldots$

Solution: Compute $(f(x+h) - f(x)/h$. Since $f(x) = x^n$

$$f(x+h) = (x+h)^n = x^n + nx^{n-1}h + h^2(\ldots)$$

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{x^n + nx^{n-1}h + h^2(\ldots) - x^n}{h} \\
&= \frac{nx^{n-1}h + h^2(\ldots)}{h} \\
&= \frac{h(nx^{n-1} + h(\ldots))}{h} = nx^{n-1} + h(\ldots)
\end{aligned}$$

Therefore the derivative is

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} nx^{n-1} + h(\ldots) = nx^{n-1}$$

Power Rule: $\frac{d}{dx} x^n = nx^{n-1}$

Special Case: $\frac{d}{dx} 1 = \frac{d}{dx} x^0 = 0$

Rules:

1. (Constant Multiple Rule) If c is a constant then

$$\frac{d}{dx}cf(x) = c\frac{d}{dx}f(x)$$

   Example: $\frac{d}{dx}7x^5 = 35x^4$

2. (Sum/Difference Rules)

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$$

Power Rule:

$$\frac{d}{dx}x^n = nx^{n-1} \quad n \text{ is any real number.}$$

Example: Differentiate $y = 5/x^3$.
So $y = 5x^{-3}$ and $y' = 15x^{-4}$
Example: Differentiate $f(x) = e^x$.
Solution:

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}e^x\frac{e^h-1}{h}$$

This says the derivative if e^x is e^x times a constant. That constant is

$$\lim_{h\to 0}\frac{e^h-1}{h}$$

and this constant is 1 by the choice of e.

$$\frac{d}{dx}e^x = e^x$$

Easy to remember!
Example: Differentiate $y = \sqrt{x} - 3e^x$
Solution: Since $y = x^{1/2} - 3e^x$,

$$y' = \frac{1}{2}x^{-1/2} - 3e^x$$