Introduction to the Derivative
Thomas Calculus Early Transcendentals §3.1, §3.2

Definition: The *derivative* of $f(x)$ at $x = a$ is defined to be

$$\frac{df}{dx}(a) = f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided the limit exists. (If the limit does not exist, f is not differentiable at a.)

Applications:

I. **Velocity:** Suppose a particle is moving in a straight line and its position at time t is given by $s(t)$. Then the velocity at time t is defined to be

$$s'(t) = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h}$$

provided the derivative exists. Observe that

$$\frac{s(t+h) - s(t)}{h}$$

is the average velocity of the particle over the time interval $[t, t+h]$. Therefore the (instantaneous) velocity is the limit of these average velocities as the time interval gets shorter and shorter

Example: A ball is tossed into the air with vertical component to its velocity 60 ft/s. The height of the ball at time t in seconds after the toss is $s(t) = 60t - 16t^2$. Find the velocity of the ball after 2 seconds. Is the ball still going up?

Solution: Compute the average velocity over various time intervals of length h starting (or ending) at time $t = 2$.

$$\begin{aligned}
\frac{s(2+h) - s(2)}{h} &= \frac{60(2+h) - 16(2+h)^2 - [60(2) - 16(2)^2]}{h} \\
&= \frac{120 + 60h - 16(4 + 4h + h^2) - [120 - 64]}{h} \\
&= \frac{120 + 60h - 64 - 64h - 16h^2) - [120 - 64]}{h} \\
&= \frac{-4h - 16h^2}{h} \\
&= \frac{h(-4 - 16h)}{h} \\
&= -4 - 16h
\end{aligned}$$

To find the velocity, take the limit of the average velocities over shorter and shorter time periods:

$$v(2) = \lim_{h \to 0} \frac{s(2+h) - s(2)}{h} = \lim_{h \to 0} -4 - 16h = -4$$

The ball is falling at 4 ft/s.

II. **Slope of a Curve:** The slope of a graph $y = f(x)$ is defined to be

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the derivative exists. Observe that the ratio

$$\frac{f(a+h) - f(a)}{h}$$

is the slope of the secant line through the two points $(a, f(a))$ and $(a+h, f(a+h))$. The slope of the curve is therefore the limiting value of the slopes of these secant lines as the two points get closer and closer.

Example: Find the slope of the curve $y = x^3 + x$ at $x = 1$. Find an equation for the tangent line there.

Solution: The slope is

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

Compute $f(1) = 2$; $f(1+h) = (1+h)^3 + 1 + h = 1 + 3h + 3h^2 + h^3 1 + h = 2 + 4h + 3h^2 + h^3$. Therefore

$$\frac{f(1+h) - f(1)}{h} = \frac{2 + 4h + 3h^2 + h^3 - 2}{h} = \frac{4h + 3h^2 + h^3}{h} = \frac{h(4 + 3h + h^2}{h} = 4+3h+h^2$$

Note the cancellation of h. Therefore

$$f'(1) = \lim_{h \to 0} 4 + 3h + h^2 = 4$$

Therefore the slope is 4. An equation for the tangent line is

$$y - 2 = 4(x - 1)$$

§3.1 (Stewart 5th ed.)

Definition: The *derivative* of $f(x)$ at $x = a$ is defined to be

$$\frac{df}{dx}(a) = f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided the limit exists. (If the limit does not exist, f is not differentiable at a.)

Alternately

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

where $x - a = h$.

Example: Find the derivative of $f(x) = 1/(x+1)$ at $x = 0, 1, 2$.

Solution: Find the derivative at a general point x

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Compute therefore $f(x+h) = 1/(x+h+1)$ and

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{1}{h}\left[\frac{1}{x+h+1} - \frac{1}{x+1}\right] \\ &= \frac{1}{h}\left[\frac{x+1}{(x+1)(x+h+1)} - \frac{x+h+1}{(x+1)(x+h+1)}\right] \\ &= \frac{1}{h}\frac{x+1-(x+h+1)}{(x+1)(x+h+1)} \\ &= \frac{1}{h}\frac{-h}{(x+1)(x+h+1)} \\ &= \frac{-1}{(x+1)(x+h+1)}\end{aligned}$$

Therefore the derivative is

$$\frac{d}{dx}\frac{1}{x+1} = \lim_{h\to 0}\frac{-1}{(x+1)(x+h+1)} = \frac{-1}{(x+1)^2}$$

Therefore $f'(0) = -1$, $f'(1) = -1/4$ and $f'(2) = -1/9$.

Section 3.2 of Stewart

Theorem 4: If f is differentiable at $x = a$ the f is continuous at $x = a$.

Proof: Write

$$\lim_{x\to a} f(x) - f(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}(x-a) = f'(a)\lim_{x\to a}(x-a) = 0$$

so that $\lim_{x\to a} f(x) = f(a)$. Since f is defined at a, this shows that f is continuous.

Example: Find the slope of the curve $y = \sqrt{x+3}$ at $x = 1$. Find an equation for the tangent line there.

Solution: The slope is

$$f'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

Compute $f(1) = 2$; $f(1+h) = \sqrt{1+h+3} = \sqrt{4+h}$. Therefore

$$\frac{f(1+h)-f(1)}{h} = \frac{\sqrt{4+h}-2}{h}$$

Rationalize the denominator.

$$\frac{\sqrt{4+h}-2}{h}\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2} = \frac{4+h-4}{h(\sqrt{4+h}+2)} = \frac{1}{\sqrt{4+h}+2}$$

Note the cancellation of h. Therefore

$$f'(1) = \lim_{h\to 0}\frac{1}{\sqrt{4+h}+2} = \frac{1}{4}$$

Therefore the slope is 1/4. An equation for the tangent line is

$$y - 2 = \frac{1}{4}(x - 1)$$

Non-Differentiable Functions: As a consequence, functions that are discontinuous at a point are not differentiable at that point. For example $f(x)\lfloor x \rfloor$ the greatest integer less or equal x is not differentiable at any integer. At other points it is differentiable.

Are there functions which are continuous that are not differentiable? Yes. For example if the curve goes straight up for an instant like $f(x) = x^{1/3}$ at $x = 0$: $x^{1/3}$ is not differentiable at $x = 0$.

Another possiblity is a function whose graph has a corner in it or a cusp. For example $f(x) = |x|$. The graph has a right angle at the origin so let's check for a derivative at $x = 0$ Consider

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

Does this limit exist? Consider

$$\lim_{h \to 0+} \frac{|h|}{h} = \lim_{h \to 0+} \frac{h}{h} = 1$$

On the other hand

$$\lim_{h \to 0-} \frac{|h|}{h} = \lim_{h \to 0+} \frac{-h}{h} = -1$$

Therefore the limit does not exist. This says that $f(x) = |x|$ is not differentiable at $x = 0$.