3.11 Differentials and Linear Approximation:

If $f(x)$ is differentiable at $x = a$ then the slope of the tangent line at $(a, f(a))$ is $f'(a)$ so that an equation for the tangent line is

$$y - f(a) = f'(a)(x - a)$$

Of course the tangent line is very close to the curve $y = f(x)$ near $(a, f(a))$ and so one can approximate $y = f(x)$ by the linearization $L(x)$ defined by

$$L(x) = f(a) + f'(a)(x - a)$$

for $x \approx a$. In fact $y = f(a)$ is known as the zero order approximation fo $y = f(x)$ whereas $y = L(x)$ is the first order approximation. (We will discuss second and higher order approximations: see Taylor and Maclaurin series later in the text.)

Example: Find the linearization of $f(x) = \sqrt{5 + 2x}$ at $a = 2$. Use the linearization to approximate $\sqrt{5 + 2(2.1)} = \sqrt{9.2}$.

Solution: We write $f(x) = (5 + 2x)^{1/2}$ and differentiate

$$f'(x) = \frac{1}{2}(5 + 2x)^{-1/2}(2) = \frac{1}{(5 + 2x)^{1/2}}$$

by the generalized power rule (or chain rule). We see that $f(2) = 3$ and $f'(2) = 1/3$ and so the linearization is

$$L(x) = 3 + \frac{1}{3}(x - 2)$$

If we set $x = 2.1$ we have $f(2.1) = \sqrt{9.2} \approx L(2.1) = 3 + (1/3)(0.1) = 3.0333\ldots$ whereas a calculator gives $f(2.1) \approx 3.0331502$.

Differentials If $y = f(x)$ the the differential dy of y is defined to be

$$dy = f'(x)\, dx$$

Example: If $y = \sin 3x$ then $y' = \cos 3x(3)$ so that

$$dy = 3 \cos 3x\, dx$$

Intuitively one thinks that, if $x = \pi/4$, for example, then a small change of "dx" in x results in a corresponding change in y of

$$dy = 3 \cos 3x\, dx = 3 \cos(3\pi/4)\, dx = -3\sqrt{2}/2\, dx \approx -2.12dx$$