Limits involving Infinity
Thomas's Calculus Early Transcendentals §2.6

Infinity as a Limit:
Example:

1. $\lim_{x\to 1+} \frac{1}{x-1} = \infty.$

2. $\lim_{x\to 1-} \frac{1}{x-1} = -\infty$ Graph $y = 1/(x-1)$

3. $\lim_{x\to 1+} \frac{1}{(x-1)^2} = \infty$

4. $\lim_{x\to 1-} \frac{1}{(x-1)^2} = \infty$

5. $\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty$

Asymptotes: An asymptote to a curve is a straight line that gets arbitrarily close to the curve far away from the origin.

Vertical Asymptotes: Recall that a vertical straight line is given by $x = c$, for example $x = -4$ or $x = 7$.

Example: Find all the vertical asymptotes of the curve

$$y = \frac{x}{x-2}$$

Solution: Look for division by 0: x=2.

Example: Find all the vertical asymptotes of the curve

$$y = \frac{2}{x^3+1}$$

Solution: Factor the bottom $x^3+1 = (x+1)(x^2-x+1)$. The quadratic is irreducible and so $x = -1$ is the only vertical asymptote.

Horizontal Asymptotes and Limits at Infinity. Recall that a horizontal line has an equation y = constant. Horizontal asymptotes are related to limits at infinity: for example if

$$\lim_{x\to\infty} f(x) = 4$$

then $y = 4$ is a horizontal asymptote (at ∞).

Examples:

1. $\lim_{x\to\pm\infty} 1/x = 0$ so that $y = 1/x$ has horizontal asymptote $y = 0$.

2. $\lim_{x\to\pm\infty} \frac{x}{x-1} = \lim_{x\to\pm\infty} \frac{x}{x}\frac{1}{1-1/x} = 1$

3. $\lim\limits_{x\to\pm\infty} \dfrac{2x^3-4x^2}{5x^3+6x+11} = \lim\limits_{x\to\pm\infty} \dfrac{x^3}{x^3}\dfrac{2x-4/x}{5+6/x^2+1/x^3} = \dfrac{2}{5}$.

4. $\lim\limits_{x\to\pm\infty} \dfrac{2x^3-4x^2}{5x^2+11} = \lim\limits_{x\to\pm\infty} \dfrac{x^2}{x^2}\dfrac{2x-4}{5+11/x^2} = \infty$ There is no horizontal asymptote.

5. $\lim\limits_{x\to\pm\infty} \dfrac{5x^2+11}{2x^3-4x^2} = \lim\limits_{x\to\pm\infty} \dfrac{x^2}{x^2}\dfrac{5+11/x^2}{2x-4} = 0$ so that $y=0$ is a horizontal asymptote.

Example: Graph $y = \frac{x}{x-2}$.
Solution:

$$\lim_{x\to 2+} \frac{x}{x-2} = \infty; \quad \lim_{x\to 2-} \frac{x}{x-2} = -\infty$$