Continuity
Thomas's Calculus Early Transcendentals§2.5

Intuitively, a function is continuous if you can graph it without lifting your pencil from the page.

Most familiar functions are continuous, such as $f(x) = x^2$, $f(x) = \sin x$, $f(x) = |x|$. But $f(x) = \tan x$ is discontinuous at $\pm\pi/2$ plus and minus all multiples of π

Definition: A function $f(x)$ is *continuous* at a if

1. a is in the domain of f.
2. $\lim_{x\to a} f(x)$ exists. (So that f is defined near a.)
3. $\lim_{x\to a} f(x) = f(a)$

The function f is continuous if it is continuous at every point of its domain.

Example (transparency for limits) Which property fails?

Examples

1. $f(x) = x^5$. f is continuous.
2. $f(x) = \dfrac{x^2}{x+1}$. f is continuous except at $x = -1$.
3. $f(x) = \dfrac{x^2-9}{x-3}$. f is continuous except at $x = 3$. Note $f(x) = x + 3$, provided $x \neq 3$.
4. $f(x) = \begin{cases} \frac{x^2-9}{x-3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 6 \end{cases}$
5. $f(x) = \cos x$. f is continuous.
6. $f(x) = \sec x$. f is continuous at all points except $x = \pm\pi/2 + k\pi$ where $k = 0, \pm1, \pm2, \pm3, \ldots$.
7. $f(x) = [|x|]$ so that $f(x)$ is the greatest integer less or equal x. f is a stair step.

Rules Suppose the $f(x)$ and $g(x)$ are continuous at $x = a$ and c is a constant. Then the following functions are continuous

1. $f(x) \pm g(x)$
2. $cf(x)$
3. $f(x)g(x)$
4. $\dfrac{f(x)}{g(x)}$ provided $g(a) \neq 0$.

Example: $\dfrac{x^4 - 3x^2 + x - 5}{(x-2)(x+1)}$ is continuous everywhere except $x = 2$ and $x = -1$.

Intermediate Value Theorem: *Suppose that $f(x)$ is a continuous function defined on an interval $a \leq x \leq b$. Then $f(x)$ takes every value between $f(a)$ and $f(b)$. In symbols if N is between $f(a)$ and $f(b)$ then there is c, $a \leq c \leq b$ so that $f(c) = N$*

Picture:

Example: If $f(x) = x^3 - 4x + 2$ then $f(0) = 2$ and $f(1) = -1$. Therefore there must be c, $0 < c < 1$ so that $f(c) = c^3 - 4c + 2 = 0$. (Here $N = 0$.)

Example: (Skip?) For what value of the constant c is the function f continuous on $-2 < x < \infty$?

$$f(x) = \begin{cases} \frac{x-3}{x^2-x-6} & \text{if } x \neq 3 \\ c & \text{if } x = 3 \end{cases}$$