Limits
Thomas Calculus Early Transcendentals §2.2 and §2.4

In our preview of calculus we saw that velocity (or instantaneous velocity) of the car at time $t = 1$ is

$$\text{“}\lim_{t\to 1}\text{”} \frac{s(t) - s(1)}{t - 1}$$

and we saw that the slope of the tangent line to a curve $y = f(x)$ at $x = 1$ is

$$\text{“}\lim_{h\to 0}\text{”} \frac{f(1 + h) - f(1)}{h}$$

Actually these two limits are the same if you replace $f \leftrightarrow s$ and $t \leftrightarrow 1 + h$ or $t - 1 \leftrightarrow h$.

We are therefore interested in looking at evaluating

$$\lim_{x\to a} F(x)$$

where a is some fixed number ($a = 0$ and $a = 1$ above) and F is some function which in the interesting cases is of the form $F(x) = f(x)/g(x)$ where $f(a) = 0$ and $g(a) = 0$.

Definition: We say that the limit of $F(x)$ as x approachs a is L

$$\lim_{x\to a} F(x) = L$$

if $|F(x) - L|$ can be made arbitrarily small by simply choosing x so that $|x - a|$ is small but $x \neq a$.

Example (Transparency and Handout)

Further Definitions: We find it worthwhile to introduce the concept of limit from the left and right: From the left is

$$\lim_{x\to a-} F(x) = L^{-}$$

if $|F(x) - L^{-}|$ can be made arbitrarily small by simply choosing x so that $|x - a|$ is small but $x < a$. On the other hand

$$\lim_{x\to a+} F(x) = L^{+}$$

if $|F(x) - L^{+}|$ can be made arbitrarily small by simply choosing x so that $|x - a|$ is small but $x > a$.

Theorem: If L^{+}, L^{-} both exist and $L^{+} = L^{-}$ then

$$\lim_{x\to a} f(x) \text{ exists and } \qquad \lim_{x\to a} f(x) = L^{+} = L^{-}$$

2.3 COMPUTING LIMITS: We have the following basic manipulation rules.

Limit Laws: Suppose

$$\lim_{x\to a} f(x) \qquad \text{and} \qquad \lim_{x\to a} g(x)$$

exist. Let c be a real constant

1. $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
2. $\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x)$
3. $\lim_{x\to a}[f(x)g(x)] = [\lim_{x\to a} f(x)][\lim_{x\to a} g(x)]$
4. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided $\lim_{x\to a} g(x) \neq 0$.

Examples:

1. $\lim_{x\to 2} x^3 - 6x + 3 = -1$
2. $\lim_{x\to -1} \dfrac{x^2 + 2x}{3x + 7} = \dfrac{-1}{4}$
3. $\lim_{x\to 2} \dfrac{x^2 - 4}{x - 2} = \lim_{x\to 2} \dfrac{(x-2)(x+2)}{x-2} = \lim_{x\to 2} x + 2 = 4.$
4. $\lim_{x\to -3} \dfrac{x+3}{x^2 + x - 6} = \lim_{x\to -3} \dfrac{x+3}{(x+3)(x-2)} = -\dfrac{1}{5}$
5. $\lim_{h\to 0} \dfrac{(1+h)^3 - 1}{h} = \lim_{h\to 0} \dfrac{1 + 3h + 3h^2 + h^3 - 1}{h} = \lim_{h\to 0} \dfrac{h(3 + 3h + h^2}{h} = 3$
6. $\lim_{x\to 1-} \dfrac{x-1}{|x-1|}$

   Here $x < 1$, so that $|x-1| = -(x-1)$ so that $\lim_{x\to 1-} \dfrac{x-1}{|x-1|} = \lim_{x\to 1-} \dfrac{x-1}{-(x-1)} = \lim_{x\to 1-} -1 = -1$

Limit Laws Continued: $\lim_{x\to a} \sqrt{f(x)} = \sqrt{\lim_{x\to a} f(x)}$

Example: $\lim_{x\to 3} \sqrt{7x^2 + 1} = 8$

Squeeze Theorem: Suppose that $f(x) \leq g(x) \leq h(x)$ and the following two limits exist and are equal:

$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x).$$

Then

$$\lim_{x\to a} g(x) = L$$

Picture:

Example: Consider $\lim_{x\to 0} x^2 \cos \frac{1}{x}$

$-1 \leq \cos\theta \leq 1$ so that

$$-x^2 \leq x^2 \cos\frac{1}{x} \leq x^2$$

Therefore

$$\lim_{x\to 0} x^2 \cos\frac{1}{x}$$

In order to find the derivative of $\sin x$ it will be necessary to compute

$$\lim_{x\to 0} \frac{\sin x}{x} \left(= \frac{0}{0}\right) =?????$$

We need to know the area of a sector of a circle of radius r^2 and opening angle θ

θ	Area
2π	πr^2
π	$\pi r^2/2$
$\pi/2$	$\pi r^2/4$
$3\pi/4$	$3\pi r^2/8$
θ	$r^2\theta/2$

Therefore the area of the sector is $r^2\theta/2$. Now recall the definition of $\sin\theta$ as the opposite over hypotheneuse. If the hypotheneuse is one then we have the picture below. Compare the areas of the inscribed and circumscribed triangles.

$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$

multiply through by 2 and divide by θ. The left inequality becomes $\sin\theta/\theta < 1$. In the right inequality, further multiply by $\cos\theta$: $\cos\theta < \sin\theta/\theta$ so that

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

Take the limit $\theta \to 0$. Now apply the Squeeze Theorem.

$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ **Examples**

1. $\lim_{\theta\to 0} \frac{\sin 2\theta}{2\theta} = 1$
2. $\lim_{\theta\to 0} \frac{\sin\theta}{2\theta} = 1/2$
3. $\lim_{\theta\to 0} \frac{\sin 2\theta}{\theta} = 2$

4. $\lim_{\theta \to 0} \frac{\sin 5\theta}{3\theta} = \frac{5}{3}$

5. $\lim_{\theta \to 0} \frac{3\theta}{\sin 5\theta} = \frac{3}{5}$

6. $\lim_{\theta \to 0} \frac{\tan \theta}{\theta} = 1$

7. $\lim_{\theta \to 0} \frac{\cos \theta}{\theta} = DNE$

8. $\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = \frac{\cos \theta - 1}{\theta} \frac{\cos \theta + 1}{\cos \theta + 1} = 0$