Preview of Differential Calculus
Thomas §2.1

We give two motivating examples for defining the derivative.

Example 1: (Velocity) Consider a small rocket on a model railway car on a long straight piece of track. We want to find the velocity of the car after the rocket booster is ignited. For example we want to find the velocity 1 second after ignition. We film the experiment with a camera that exposes a 32 frames of film every second. Behind the track is a backdrop with the distance along the track marked in feet.

t (time in seconds)	1	2	3/2	5/4	9/8	17/16	33/32
s (distance in ft)	28.0	145.0	69.4	46.2	36.8	32.35	30.15

and of course there is also data on the time before $t = 1$

t (time in seconds)	1	0	1/2	5/4	9/8	15/16	31/32
s (distance in ft)	28.0	0.0	9.0	17.2	20.6	24.0	25.9

From these numbers we can calculate the average velocity of the car over the time intervals beginning at time $t = 1$

Time interval (s)	Average velocity (ft/s)
$1 < t < 2$	(145.0-28.0)/(2-1)=117
$1 < t < 3/2$	(69.4-28.0)/(3/2-1) =82.8
$1 < t < 5/4$	(46.2-28.0)/(5/4-1) = 72.8
$1 < t < 9/8$	(36.8-28.0)/(9/8-1) = 70.4
$1 < t < 17/16$	(32.35-28.0)/(17/16-1)= 69.6
$1 < t < 33/32$	(30.15-28.0)/(33/32 - 1)=68.8
$0 < t < 1$	(0.0-28.0)/(0-1)=28.0
$1/2 < t < 1$	(9.0-28.0)/(1/2-1)= 38.0
$3/4 < t < 1$	(17.2-28.0)/(3/4-1)= 51.2
$7/8 < t < 1$	(20.6-28.0)/(7/8-1)= 59.2
$15/16 < t < 1$	(24.0-28.0)/(15/16-1)= 64.0
$31/32 < t < 1$	(25.9-28.0)/(31/32-1)= 67.2

Newton defined the (instantaneous) velocity at time t_0 to be

$$``\lim_{t_1 \to t_0}" \frac{s(t_1) - s(t_0)}{t_1 - t_0}$$

(where lim is read "the limit as t_1 goes to t_0") The data here suggests that the instantanous velocity is about 68 ft/s. Of course this estimate only uses the data at the three times t=31/32,1,33/32. We'd prefer data even closer to $t = 1$ but that data may be physically difficult to obtain.

Before proceeding to the second motivational example observe that Newton's definition can be written equivalently as

$$``\lim_{h \to 0}" \frac{s(t_0 + h) - s(t_0)}{h}$$

by setting $h = t_1 - t_0$ so that $t_0 + h = t_1$.

Example 1: (Slope of a Curve) The slope of a curve $y = f(x)$ at a point $(a, f(a))$ is the slope of the tangent line to the curve there. It is denoted $f'(a)$ and is the derivative of $f(x)$ at $x = a$. The tangent line is the line of best fit to the curve at the point. It is unique provided it exists. To find the slope $f'(a)$ of the curve $y = f(x)$, we approximate the tangent line by secant lines that cross the curve twice, once at the point $(a, f(a))$ and a second time at a "nearby" point.

Example: Find the derivative of $f(x) = x^2 + x$ at $x = 1$.

Solution: Draw a picture.

Find the slope of some secant lines that pass through (1,2) and a nearby point also on the curve $y = x^2 + x$. For example when $x = 2$, $y = 6$ so that the slope of the secant is

$$\frac{6-2}{2-1} = 4$$

Similarly when $x = 0$, $y = 0$ so that the slope is

$$\frac{0-2}{0-1} = 2.$$

Let us check points very nearby. Consider $x = 1 + h$ where h is small. Then $y = (1+h)^2 + 1 + h = 1 + 2h + h^2 + 1 + h$. The slope is

$$\frac{1 + 2h + h^2 + 1 + h - 2}{1 + h - 1} = \frac{3h + h^2}{h} = 3 + h$$

or in symbols the slope is

$$\frac{f(1+h) - f(1)}{h}$$

For the slope of the tangent let the two points get closer and closer:

$$\text{“}\lim_{h\to 0}\text{”}\frac{f(1+h) - f(1)}{h} = \text{“}\lim_{h\to 0}\text{”}3 + h = 3$$

The slope of the tangent line is 3