1 10.5 Lines and Planes

Examples: What subset of $\mathbb{R}^3$ does $z = 0$ describe? The xy-plane. What about the equation $3x - 2y + z = 0$? Observe that we can write the equation as

$$\langle x, y, z\rangle \cdot \langle 3, -2, 1\rangle = 0$$

This is the set of all points that are perpendicular to the vector $\langle 3, -2, 1\rangle$. This is a plane and $\langle 3, -2, 1\rangle$ is the *normal* to the plane.

Example: Find an equation for the plane through (1,5,-3) which is perpendicular to $\langle 3, -2, 1\rangle$

$$3(x-1) - 2(y-5) + (z+3) = 0 \quad \text{or} \quad 3x - 2y + z = -10$$

Planes: The plane through (x_0, y_0, z_0) with normal $\langle a, b, c\rangle$ has equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

By the same reasoning $ax+by+cz = d$ is the equation of a plane with normal $\langle a, b, c\rangle$

Example: The plane $-2x + 7y - 4z = 12$ is a plane with normal $\langle -2, 7, -4\rangle$. Find a point on the plane. (3,2,-1).

Example Find an equation for the plane that contains the three points $P(-2, 1, 4)$, $Q(3, 2, -3)$ and $R(1, -1, 4)$.

Solution: How do we find a normal to the plane. $\overrightarrow{PQ} = \langle 5, 1, -7\rangle$ and $\overrightarrow{PR} = \langle 3, -2, 0\rangle$ and a normal is

$$\begin{aligned} \overrightarrow{PQ} \times \overrightarrow{PR} &= \det \begin{bmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 5 & 1 & -7 \\ 3 & -2 & 0 \end{bmatrix} \\ &= (0 - 14)\overrightarrow{i} - (0 + 21)\overrightarrow{j} + (-10 - 3)\overrightarrow{k} \\ &= -14\overrightarrow{i} - 21\overrightarrow{j} - 13\overrightarrow{k} \end{aligned}$$

A normal to the plane is $\langle -14, -21, -13\rangle$. Any multiple of this vector is also a normal: $\langle 14, 21, 13\rangle$ for example. An equation of the plane is therefore

$$14(x+2) + 21(y-1) + 13(z-4) = 0 \quad \text{or} \quad 14x + 21y + 13z = 45$$

Alternatively you could find the unknown d in the equation $14x+21y+13z = d$ by plugging points. Check your answer by plugging points P, Q and R.

Definition: Two planes are said to be parallel if their normals are parallel. The angle between planes is defined to be the angle between their normals.

Parallel planes which are not the same do not intersect. $14x+21y+13z = 11$ and $-28x - 42y - 26z = \sqrt{2}$ are parallel but have no intersection. The planes $2x + y - 3z = 11$ and $-5x + 4z = 2$ are not parallel and the angle between them is θ where $0 < \theta \leq \pi$ and

$$\begin{aligned}\cos\theta &= \frac{\langle 2,1,-3\rangle \cdot \langle -5,0,4\rangle}{|\langle 2,1,-3\rangle||\langle -5,0,4\rangle|} = \frac{-22}{\sqrt{14}\sqrt{41}} \\ &\approx -0.9182622582426504073980129486190978506214017901329 1\end{aligned}$$

$\theta \approx 2.734465697573565193333137626198359004594991354408 1.$

Example: Find the distance from the point $P(2,1,-1)$ to the plane $4x - y + 2z = 3$.

Solution: Pick a point on the plane. For example $Q(0,-3,0)$ and form the vector $\overrightarrow{PQ} = \langle -2,-4,-1\rangle$ We want the component of $\overrightarrow{PQ}$ along the normal $\overrightarrow{n} = \langle 4,-1,2\rangle$. Recall from Section 10.3

$$\text{comp}_{\overrightarrow{n}}\overrightarrow{PQ} = \frac{\overrightarrow{n}\cdot\overrightarrow{PQ}}{|\overrightarrow{n}|}(= |\overrightarrow{PQ}|\cos\theta) = \frac{-6}{\sqrt{21}} = -\frac{2}{7}\sqrt{21}$$

The distance from P to the plane $4x - y + 2z = 3$ is $|-2\sqrt{21}/7| = 2\sqrt{21}/7$.

In general, the distance from $P(x_1, y_1, z_1)$ to the plane $ax+by+cz+d = 0$ is

$$\left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2+b^2+c^2}}\right|$$

This can be verified as in the example by choosing $Q(x_2, y_2, z_2)$ on the plane (so that $ax_2 + by_2 + cz_2 + d = 0$) and $\overrightarrow{n} = \langle a,b,c\rangle$

$$\begin{aligned}\text{comp}_{\overrightarrow{n}}\overrightarrow{PQ} &= \frac{\overrightarrow{n}\cdot\overrightarrow{PQ}}{|\overrightarrow{n}|} = \frac{\langle a,b,c\rangle\cdot\langle x_2-x_1, y_2-y_1, z_2-z_1\rangle}{\sqrt{a^2+b^2+c^2}} \\ &= \frac{ax_2+by_2+cz_2-(ax_1+by_1+cz_1)}{\sqrt{a^2+b^2+c^2}} = -\frac{d+ax_1+by_1+cz_1}{\sqrt{a^2+b^2+c^2}}\end{aligned}$$

and now we take absolute value to get the result.