1 10.2 Cross Product

If $\overrightarrow{a} = \langle a_1, a_2, a_3 \rangle = a_1 \overrightarrow{i} + a_2 \overrightarrow{j} + a_3 \overrightarrow{k}$ and $\overrightarrow{b} = \langle b_1, b_2, b_3 \rangle$ then

$$\overrightarrow{a} \times \overrightarrow{b} = \langle a_2b_3 - a_2b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

Mnemonic: The determinant can be helpful to remember the formula for $\overrightarrow{a} \times \overrightarrow{b}$

$$\begin{aligned}
\overrightarrow{a} \times \overrightarrow{b} &= \begin{bmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \\
&= \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} \overrightarrow{i} - \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} \overrightarrow{j} \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \overrightarrow{k}
\end{aligned}$$

Theorem For any $\overrightarrow{a}$ $\overrightarrow{b}$ with angle θ between them. $|\overrightarrow{a} \times \overrightarrow{b}| = |\overrightarrow{a}||\overrightarrow{b}| \sin\theta$

Proof. We will show

$$|\overrightarrow{a} \times \overrightarrow{b}|^2 + (\overrightarrow{a} \cdot \overrightarrow{b})^2 = |\overrightarrow{a}|^2 |\overrightarrow{b}|^2$$

and this will imply that

$$\begin{aligned}
|\overrightarrow{a} \times \overrightarrow{b}|^2 &= |\overrightarrow{a}|^2|\overrightarrow{b}|^2 - (\overrightarrow{a} \cdot \overrightarrow{b})^2 = |\overrightarrow{a}|^2|\overrightarrow{b}|^2[1 - (\cos\theta)^2] \\
&= |\overrightarrow{a}|^2|\overrightarrow{b}|^2(sin\theta)^2
\end{aligned}$$

and this will complete the proof since $\sin\theta \geq 0$. We need only prove the first identity therefore. We compute

$$\begin{aligned}
&|\overrightarrow{a} \times \overrightarrow{b}|^2 + (\overrightarrow{a} \cdot \overrightarrow{b})^2 \\
&= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 + (a_1b_1 + a_2b_2 + a_3b_3)^2 \\
&= [a_2^2b_3^2 + a_3^2b_2^2 + a_3^2b_1^2 + a_1^2b_3^2 + a_1^2b_2^2 + a_2^2b_1^2] - 2(a_2a_3b_2b_3 + a_1a_3b_1b_3 \\
&\quad + a_1a_2b_1b_2) + [a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2] + 2(a_1a_2b_1b_2 + a_1a_3b_1b_3 + a_2a_3b_2b_3) \\
&= a_2^2b_3^2 + a_3^2b_2^2 + a_3^2b_1^2 + a_1^2b_3^2 + a_1^2b_2^2 + a_2^2b_1^2 + a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2
\end{aligned}$$

because the terms in parentheses cancel. On the other hand

$$\begin{aligned}
|\overrightarrow{a}|^2|\overrightarrow{b}|^2 &= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) \\
&= a_1^2b_1^2 + a_1^2b_2^2 + a_1^2b_3^2 + a_2^2b_1^2 + a_2^2b_2^2 + a_2^2b_3^2 + a_3^2b_1^2 + a_3^2b_2^2 + a_3^2b_3^2
\end{aligned}$$

Comparing the two expressions we see the initial identity is verified □

Consequences:

1. Two non zero vectors $\overrightarrow{a}$, $\overrightarrow{b}$ are parallel if and only if $\overrightarrow{a} \times \overrightarrow{b} = \overrightarrow{0}$.
2. The parallelogram determined by two vectors $\overrightarrow{a}$, $\overrightarrow{b}$ has area $|\overrightarrow{a} \times \overrightarrow{b}|$
3. The triangle determined by two vectors $\overrightarrow{a}$, $\overrightarrow{b}$ has area $(1/2)|\overrightarrow{a} \times \overrightarrow{b}|$.

Properties If $\overrightarrow{a}$ $\overrightarrow{b}$ $\overrightarrow{c}$ are vectors and c is a real number then

1. $\overrightarrow{a} \times \overrightarrow{b} = -\overrightarrow{b} \times \overrightarrow{a}$
2. $\overrightarrow{a} \times (\overrightarrow{b} + \overrightarrow{c}) = \overrightarrow{a} \cdot \overrightarrow{b} + \overrightarrow{a} \cdot \overrightarrow{c}$ (Distributive Property)
3. $(c\overrightarrow{a}) \cdot \overrightarrow{b} = c(\overrightarrow{a} \cdot \overrightarrow{b})$
4. $\overrightarrow{a} \cdot \overrightarrow{0} = 0$

These properties follow directly form the definition.

Often in the physical sciences the dot product is defined in a different way which has a certain physical appeal.

Theorem Let θ be angle between two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$. Then

$$\overrightarrow{a} \cdot \overrightarrow{b} = |\overrightarrow{a}||\overrightarrow{b}|\cos\theta$$

The angle between vectors is the angle at the vertex of the triangle determined by placing the two vectors tail to tail. It is understood that $0 \leq \theta \leq \pi$. If $\theta = 0$ the vectors must be parallel and "in the same direction" and if $\theta = \pi$ the vectors are parallel but in the opposite direction. If $\theta = \pi/2$ then teh vectors are perpendicular.

Proof. Recall the cosine law from trigonometry. $a^2 + b^2 - 2ab\cos\theta = c^2$ Converting to vector notation

$$\overrightarrow{a} \cdot \overrightarrow{a} + \overrightarrow{b} \cdot \overrightarrow{b} - 2|\overrightarrow{a}||\overrightarrow{b}|\cos\theta = |\overrightarrow{a} - \overrightarrow{b}|^2 = \overrightarrow{a} \cdot \overrightarrow{a} + \overrightarrow{b} \cdot \overrightarrow{b} - 2\overrightarrow{a} \cdot \overrightarrow{b}$$

Simplifying we have the result. $\square$

Observe that the dot product provides an easy way to compute the angle between two vectors

$$\cos\theta = \frac{\overrightarrow{a} \cdot \overrightarrow{b}}{|\overrightarrow{a}||\overrightarrow{b}|}$$

Definition: Two vectors $\overrightarrow{a}$, $\overrightarrow{b}$ are said to be orthogonal or perpendicular if $\overrightarrow{a} \cdot \overrightarrow{b} = 0$.

Orthgonal is equivalent to $\theta = \pi/2$.

Example: Let $\overrightarrow{a} = \langle -3, 5, -2\rangle$ and $\overrightarrow{b} = \langle 4, 4, ?\rangle$ then $\overrightarrow{a}$ and $\overrightarrow{b}$ are orthogonal.

textbfProjections: Given two vectors **Example** If $\overrightarrow{a}$ and $\overrightarrow{b}$ are two vectors then we define the *component of* $\overrightarrow{b}$ *along* $\overrightarrow{a}$ by

$$\text{comp}_a \overrightarrow{b} = \frac{\overrightarrow{a} \cdot \overrightarrow{b}}{|\overrightarrow{a}|}$$

which is the "length" of the vector that we get by "dropping a perpendicular" from $\overrightarrow{b}$ onto the line determined by $\overrightarrow{a}$. This vector is the *projection of* $\overrightarrow{b}$ *onto* $\overrightarrow{a}$ and we have

$$\text{proj}_a \overrightarrow{b} = \frac{\overrightarrow{a} \cdot \overrightarrow{b}}{|\overrightarrow{a}|^2}\overrightarrow{a}$$

Observe that if the projection and $\overrightarrow{a}$ are in opposite directions then $\text{comp}_a \overrightarrow{b} < 0$. In general $\text{proj}_a \overrightarrow{b}$ is parallel to $\overrightarrow{a}$. (Not $\overrightarrow{b}$). See the picture below.

Example: If $\overrightarrow{a} = \langle 1, -3, 5\rangle$ and $\overrightarrow{b} = \langle -2, 0, 4\rangle$ then

$$\text{comp}_a \overrightarrow{b} = \frac{\overrightarrow{a} \cdot \overrightarrow{b}}{|\overrightarrow{a}|} = \frac{18}{\sqrt{35}}$$

and

$$\text{proj}_a \overrightarrow{b} = \frac{\overrightarrow{a} \cdot \overrightarrow{b}}{|\overrightarrow{a}|^2}\overrightarrow{a}\frac{18}{35}\langle 1, -3, 5\rangle$$

Example (Work): Bartholomew is playing shuffle board and he exerts a force $\langle 4, 2\rangle$ Newtons on the disk which he pushes $\langle 2/3, 0\rangle$ meters (the displacement) before releasing it. Bartholomew does $\langle 4, 2\rangle \cdot \langle 2/3, 0\rangle = 8/3$ Newton meters work on the disk.