1 12.3 Dot Product

If $\overrightarrow{a} = \langle a_1, a_2, a_3 \rangle$ and $\overrightarrow{b} = \langle b_1, b_2, b_3 \rangle$ then

$$\overrightarrow{a} \cdot \overrightarrow{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Properties If $\overrightarrow{a}$ $\overrightarrow{b}$ $\overrightarrow{c}$ are vectors and c is a real number then

1. $\overrightarrow{a} \cdot \overrightarrow{a} = |\overrightarrow{a}|^2 = a_1^2 + a_2^2 + a_3^2$
2. $\overrightarrow{a} \cdot \overrightarrow{b} = \overrightarrow{b} \cdot \overrightarrow{a}$ (Commutative Property)
3. $\overrightarrow{a}(\overrightarrow{b} + \overrightarrow{c}) = \overrightarrow{a} \cdot \overrightarrow{b} + \overrightarrow{a} \cdot \overrightarrow{c}$ (Distributive Property)
4. $(c\overrightarrow{a}) \cdot \overrightarrow{b} = c(\overrightarrow{a} \cdot \overrightarrow{b})$
5. $\overrightarrow{a} \cdot \overrightarrow{0} = 0$

These properties follow directly form the definition.

Often in the physical sciences the dot product is defined in a different way which has a certain physical appeal.

Theorem Let θ be angle between two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$. Then

$$\overrightarrow{a} \cdot \overrightarrow{b} = |\overrightarrow{a}||\overrightarrow{b}| \cos\theta$$

The angle between vectors is the angle at the vertex of the triangle determined by placing the two vectors tail to tail. It is understood that $0 \le \theta \le \pi$. If $\theta = 0$ the vectors must be parallel and "in the same direction" and if $\theta = \pi$ the vectors are parallel but in the opposite direction. If $\theta = \pi/2$ then teh vectors are perpendicular.

Proof. Recall the cosine law from trigonometry. $a^2 + b^2 - 2ab\cos\theta = c^2$ Converting to vector notation

$$\overrightarrow{a} \cdot \overrightarrow{a} + \overrightarrow{b} \cdot \overrightarrow{b} - 2|\overrightarrow{a}||\overrightarrow{b}| \cos\theta = |\overrightarrow{a} - \overrightarrow{b}|^2 = \overrightarrow{a} \cdot \overrightarrow{a} + \overrightarrow{b} \cdot \overrightarrow{b} - 2\overrightarrow{a} \cdot \overrightarrow{b}$$

Simplifying we have the result. □

Observe that the dot product provides an easy way to compute the angle between two vectors

$$\cos\theta = \frac{\overrightarrow{a}\cdot\overrightarrow{b}}{|\overrightarrow{a}||\overrightarrow{b}|}$$

Definition: Two vectors $\overrightarrow{a}$, $\overrightarrow{b}$ are said to be orthogonal or perpendicular if $\overrightarrow{a}\cdot\overrightarrow{b} = 0$.

Orthgonal is equivalent to $\theta = \pi/2$.

Example: Let $\overrightarrow{a} = \langle -3, 5, -2\rangle$ and $\overrightarrow{b} = \langle 4, 4, ?\rangle$ then $\overrightarrow{a}$ and $\overrightarrow{b}$ are orthogonal.

textbfProjections: Given two vectors **Example** If $\overrightarrow{a}$ and $\overrightarrow{b}$ are two vectors then we define the *component of* $\overrightarrow{b}$ *along* $\overrightarrow{a}$ by

$$\text{comp}_a\overrightarrow{b} = \frac{\overrightarrow{a}\cdot\overrightarrow{b}}{|\overrightarrow{a}|}$$

which is the "length" of the vector that we get by "dropping a perpendicular" from $\overrightarrow{b}$ onto the line determined by $\overrightarrow{a}$. This vector is the *projection of* $\overrightarrow{b}$ *onto* $\overrightarrow{a}$ and we have

$$\text{proj}_a\overrightarrow{b} = \frac{\overrightarrow{a}\cdot\overrightarrow{b}}{|\overrightarrow{a}|^2}\overrightarrow{a}$$

Observe that if the projection and $\overrightarrow{a}$ are in opposite directions then $\text{comp}_a\overrightarrow{b} < 0$. In general $\text{proj}_a\overrightarrow{b}$ is parallel to $\overrightarrow{a}$. (Not $\overrightarrow{b}$). See the picture below.

Example: If $\overrightarrow{a} = \langle 1, -3, 5\rangle$ and $\overrightarrow{b} = \langle -2, 0, 4\rangle$ then

$$\text{comp}_a\overrightarrow{b} = \frac{\overrightarrow{a}\cdot\overrightarrow{b}}{|\overrightarrow{a}|} = \frac{18}{\sqrt{35}}$$

and

$$\text{proj}_a\overrightarrow{b} = \frac{\overrightarrow{a}\cdot\overrightarrow{b}}{|\overrightarrow{a}|^2}\overrightarrow{a}\frac{18}{35}\langle 1, -3, 5\rangle$$