1 12.2 Vectors

A vector has both magnitude and direction. It can be thought of as an arrow $\overrightarrow{PQ}$ going from a point $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$. However it is not the points P and Q that are important but the direction and magnitude: $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$. For example the vector $\overrightarrow{PQ}$ is equal to $\overrightarrow{RS}$ if for $P(1, 1, 1)$ $Q(2, 3, 4)$ and $R(-2, -3, 0)$ and $S(-1, -1, 3)$. Therefore the vector $\overrightarrow{PQ}$ can be identified with $\langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$ an ordered triple of real numbers. For example, for $P(1, 1, 1)$ and $Q(2, 3, 4)$ the vector is $\mathbf{a} = \overrightarrow{a} = \langle 1, 2, 3 \rangle$ and it is the same vector no matter where it starts so that $P(x_1, y_1, z_1)$ but $\overrightarrow{PQ} = \overrightarrow{a}$ if and only if $Q = Q(x_1 + 1, x_2 + 2, x_3 + 3)$. Vectors may also be considered in $\mathbb{R}^2$ (the xy-plane) not just $\mathbb{R}^3$.

Applications:

1. Displacement vectors: If an object is moved 3 units in the (positive) x direction; 2 units in the y direction and one unit down then the displacement is the vector $\langle 3, 2, -1 \rangle$.

2. Velocity vectors: If a kite is moving 4 units/minute in the (positive) x direction; 2 units/minute in the y direction and half a unit per minute down then the velocity vector is $\langle 4, 2, -0.5 \rangle$.

3. Force and acceleration:. If a 5 kg mass is accelerated at 4 m/s in the positive x-direction and -2 m/s in the y direction and 3 m/s up then the acceleration vector is $\langle 4, -2, 3 \rangle$ and the force is $5\langle 4, -2, 3 \rangle = \langle 20, -10, 15 \rangle$ Newtons.

4. Position Vectors. There is a one to one correspondance between points $P(x, y, z)$ in $\mathbb{R}^3$ and vectors $\langle x, y, z \rangle$ that start at the origin and end at the point P. This is not a physical application but it gives a quick mental picture.

Addition of Vectors Given two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ which we think of as arrows or directed line segments we can add the vectors $\overrightarrow{a} + \overrightarrow{b}$ by placing the tail of one on the head of the other and join the tail of the first to the head of the second. Alternately the vectors can be thought of as the edges of a rectangle (in 3 space) and their sum is one diagonal. Diagram:

Scalar multiplication of Vectors If $\overrightarrow{a}$ is a vector and c is a real number then $c\overrightarrow{a}$ is the vector parallel to $\overrightarrow{a}$ and of length $|c|$ times the length of $\overrightarrow{a}$. Diagram:

If $c < 0$ then $c\overrightarrow{a}$ is in the opposite direction to $\overrightarrow{a}$ Notice that $-\overrightarrow{a}$ is the same magnitude but opposite direction as $\overrightarrow{a}$.

Subtraction of vectors: The other diagonal.

Components Let $\overrightarrow{a} = \langle a_1, a_2, a_3\rangle$, $\overrightarrow{b} = \langle b_1, b_2, b_3\rangle$ and c is a real number:

$$\begin{aligned}
\overrightarrow{a} + \overrightarrow{b} &= \langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle \\
\overrightarrow{a} - \overrightarrow{b} &= \langle a_1 - b_1, a_2 - b_2, a_3 - b_3\rangle \\
c\overrightarrow{a} &= \langle ca_1, ca_2, ca_3\rangle
\end{aligned}$$

Length: The length of $\overrightarrow{a} = \langle a_1, a_2, a_3\rangle$ is

$$|\overrightarrow{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Therefore $|\overrightarrow{a}|$ corresponds to the distance between P and Q if $\overrightarrow{PQ} = \overrightarrow{a}$.
We say that $\overrightarrow{a}$ is a **unit** vector if its length is $|\overrightarrow{a}| = 1$.

Special Unit Vectors The standard basis vectors are defined by

$$\overrightarrow{i} = \langle 1, 0, 0\rangle, \quad \overrightarrow{j} = \langle 0, 1, 0\rangle, \quad \overrightarrow{k} = \langle 0, 0, 1\rangle$$

Physicists sometimes use the notation $\hat{\imath}$,$\hat{\jmath}$, $\hat{k}$. We may write

$$\langle a_1, a_2, a_3\rangle = a_1\overrightarrow{i} + a_2\overrightarrow{j} + a_3\overrightarrow{k}$$

so that any vector can be written as a linear combination of the three standard basis vectors.

Example: Find a unit vector in the opposite direction to $\langle -3, 4, 7\rangle$

Solution: A unit vector in the same direction as $\langle -3, 4, 7\rangle$ is given by $(1/\sqrt{74})\langle -3, 4, 7\rangle$ and so the unit vector in the opposite direction is $-(1/\sqrt{74})\langle -3, 4, 7\rangle$.

Example: A pilot wishes to fly due north but the wind is blowing steadily at 70 knots from the southwest. If the pilot maintains an air speed of 250 knots then what direction should the pilot fly?

Solution:Let $\overrightarrow{j} = \langle 0, 1\rangle$ be north. Then the unit vector in the northeast direction is $(\overrightarrow{i} + \overrightarrow{j})/\sqrt{2}$. We have

$$70(\overrightarrow{i} + \overrightarrow{j})/\sqrt{2} + 250(a_1\overrightarrow{i} + a_2\overrightarrow{j}) = |v|\overrightarrow{j}$$

where $a_1\overrightarrow{i} + a_2\overrightarrow{j})$ is the unit vector in the direction that the pilot will fly and $|v|$ will be the groundspeed of the plane. Two equations in two unknowns but not linear. Solve: $a_1 = -70/250\sqrt{2} = -7/25\sqrt{2}$. $a_2^2 = 1 - a_1^2 = 1 - 49/1250 = 1201/1250$. The pilot should fly in the the direction

$$a_1\overrightarrow{i} + a_2\overrightarrow{j} = \frac{1}{25\sqrt{2}}(-7\overrightarrow{i} + \sqrt{1201}\overrightarrow{j})$$

We could also determine the groundspeed $|v| = 70\sqrt{2} + \dfrac{10\sqrt{1201}}{\sqrt{2}} \approx 294.5485$ (using a caculator).