**11.5 Area in Polar Coordinates:**

Find the area enclosed by a curve given in polar coordinates by $r = f(\theta)$. For example find the area of one leaf of the rose $r = a\sin 3\theta$. The first step is to derive the formula for the area of a circular sector of radius $r$ and opening angle $\theta$. This is a portion of a circular disk. The picture is

| $\theta$ | Area |
|---|---|
| $2\pi$ | $\pi r^2$ |
| $\pi$ | $\pi r^2/2$ |
| $\pi/2$ | $\pi r^2/4$ |
| $\theta$ | $\theta r^2/2$ |
| $\vdots$ | $\vdots$ |

Area of a sector of radius $r$ and opening angle $\theta$: Area $= r^2\theta/2$.

Next approximate the area inside the curve $r = f(\theta)$ by sectors. See picture.

The area is approximately

$$\sum_{i=1}^{n} \frac{1}{2} f(\theta_i)^2 \Delta\theta$$

or in the limit as $\Delta\theta \to 0$ and $n \to \infty$.

$$\int_{\alpha}^{\beta} \frac{1}{2} f(\theta)^2 \, d\theta$$

Example: Find the area of one leaf of the rose $r = \sin 3\theta$.

Solution: A leaf is traced out for $0 \le \theta \le \pi/3$ by our earlier work. Therefore the area is

$$\int_0^{\pi/3} \frac{1}{2}(\sin 3\theta)^2 \, d\theta = \frac{1}{4}\int_0^{\pi/3} (1 - \cos 6\theta) \, d\theta = \frac{1}{4}[\theta - \frac{1}{6}\sin 6\theta]_0^{\pi/3} = \pi/12$$

Example: Find the area inside the cardioid $r = 1 + \sin\theta$ but outside the circle $r = 3/2$.

Solution: Sketch! The two curves intersect when $3/2 = 1 + \sin\theta$ or $\theta = \pi/6$ to $\theta = 5\pi/6$. Therefore area is

$$\begin{aligned}
\int_{\pi/6}^{5\pi/6} \frac{1}{2}(1+\sin\theta)^2 - \frac{1}{2}9/4\,d\theta &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} 1 + 2\sin\theta + (\sin\theta)^2 - 9/4\,d\theta \\
&= [-\frac{5}{8}\theta - \cos\theta]_{\pi/6}^{5\pi/6} + \frac{1}{4}\int 1 - \cos 2\theta\,d\theta \\
&= [-\frac{3}{8}\theta - \cos\theta - \frac{1}{8}\sin 2\theta]_{\pi/6}^{5\pi/6} \\
&= -\frac{\pi}{4} - \cos 5\pi/6 + \cos\pi/6 - \frac{1}{8}(\sin 5\pi/3 - \sin\pi/3) \\
&= -\frac{\pi}{4} + \sqrt{3} + \sqrt{3}/8
\end{aligned}$$

Example: Express the area common to the two circles $r = \cos\theta$ and $r = \sin\theta$ in terms of one or more integrals. Do not evaluate.

Solution. Graph.

$$2\int_0^{\pi/4} \frac{1}{2}(\cos\theta)^2\,d\theta = \frac{1}{2}\int_0^{\pi/4} 1 + \cos 2\theta\,d\theta = \frac{\pi}{8} + \frac{1}{4}$$