11.2 Parametric Curves 2: Areas and Tangents

Tangent Lines: A particle is moving along a curve $(x(t), y(t))$. Fix $t = t_0$. Then $x'(t_0)$ represents how far the particle will travel in the x-direction in one time unit (second) if there is no force. Similarly $y'(t_0)$ is how far the particle will travel in the y-direction in one second. Therefore if forces are turned off then the particle will drift from $(x(t_0), y(t_0))$ to $(x(t_0) + x'(t_0), y(t_0) + y'(t_0))$. Picture

The slope of the tangent line to the curve should be

$$\frac{y'(t_0)}{x'(t_0)}$$

Check this. Assume that it is possible to eliminate t and solve for y in terms of x. Then the chain rule says

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

Solve for dy/dx: the slope is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t_0)}{x'(t_0)}$$

as we thought.

Example: Find dy/dx and d^2y/dx^2 at $(\sqrt{2}, 3\sqrt{2}/2)$ if

$$x(t) = 2\cos t, \qquad y(t) = 3\sin t$$

and an equation for the tangent line to the curve.

Solution. $dx/dt = -2\sin t$ and $dy/dt = 3\cos t$. At $(\sqrt{2}, 3\sqrt{2}/2)$, $t = \pi/4$ so that $dx/dt = -\sqrt{2}$ and $dy/dt = 3\sqrt{2}/2)$. The slope of the tangent line is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{3\cos t}{2\sin t} = -\frac{3\sqrt{2}/2}{\sqrt{2}} = -3/2$$

(Note we leave dy/dx as a function of t.) An equation for the tangent line is:

$$y - 3\sqrt{2}/2 = -\frac{3}{2}(x - \sqrt{2}) \qquad \text{or} \qquad y = -\frac{3}{2}x + 3\sqrt{2}$$

Next find d^2y/dx^2. By the chain rule

$$\frac{df}{dx} = \frac{df/dt}{dx/dt}.$$

Applying this with $f = dy/dx = -\frac{3\cos t}{2\sin t} = -\frac{3}{2}\cot t$ we have

$$\frac{d^2 f}{dx^2} = -\frac{3}{2}\frac{\frac{d\cot t}{dt}}{-2\sin t} = -\frac{3}{4}(\csc t)^3$$

At $(\sqrt{2}, 3\sqrt{2}/2)$ where $t = \pi/4$, $d^2y/dx^2 = 3\sqrt{2}/16$.

Parametric Curves: Length and Surface Area:

The length of a curve $x(t)$ and $y(t)$, $a \le x \le b$. The distance traveled by a particle in the time interval Δt is

$$\sqrt{x'(t_i)^2 + y'(t_i)^2}\Delta t.$$

The total distance traveled is

$$\int_a^b \sqrt{x'(t_i)^2 + y'(t_i)^2}\, dt$$

Example: Find the length of the curve $x = t^3$ and $y = t^2$, $0 \le t \le 3$.

Solution: Here $x' = 3t^2$ and $y' = 2t$ so that the length is

$$\int_0^3 \sqrt{(3t^2)^2 + (2t)^2}\, dt = \int_0^3 (9t^4 + 4t^2)^{1/2}\, dt.$$

Factor out a t and substitute $u = 9t^2 + 4$, so that $du = 9tdt$. The length is therefore

$$\int_0^3 (9t^2+4)^{1/2} t\, dt = \frac{1}{9}\int_{t=0}^{t=3} u^{1/2}\, du = \frac{1}{9}\frac{2}{3}u^{3/2}|_{t=0}^{t=3} = \frac{2}{27}(9t^2+4)^{3/2}|_0^3 = \frac{2}{27}(85^{3/2} - 8)$$

Example. Find the distance traveled by a particle traveling along the path $x(t) = 5\cos t$, $y = 5\sin t$, $0 \le t \le 6\pi$. Compare to the length of the curve.

Solution. The circle has length (or circumference) $2\pi r = 10\pi$. The distance traveled is 30π.

Surfaces of Rotation. If the curve $(x(t), y(t))$, $a \le t \le b$ is rotated around the x-axis then the surface area is

$$\int_a^b 2\pi y(t)(x'(t)^2 + y'(t)^2)^{1/2}\, dt$$

and around the y-axis then the surface area is

$$\int_a^b 2\pi x(t)(x'(t)^2 + y'(t)^2)^{1/2}\, dt$$

Example: Find the surface area of a sphere of radius $a > 0$.

Solution: The surface is traced out by rotating $x(t) = a\cos t$ $y(t) = a\sin t$ around the x-axis or y-axis. Let's say the x-axis. Then $0 \le t \le \pi$. Since $x'(t) = -a\sin t$ and $y'(t) = a\cos t$ so that $[x'(t)^2 + y'(t)^2]^{1/2} = [a^2(\cos t)^2 + a^2(\sin t)^2]^{1/2} = a$. Therefore the surface area is

$$2\pi \int_0^\pi a\sin t\, a\, dt = 2\pi a^2 \int_0^\pi \sin t\, dt = 4\pi a^2$$