10.7 Expanding Functions as Power Series:

Example: Recall the geometric series:

$$\frac{a}{1-x} = a + ax + ax^2 + ax^3 + ax^4 + \ldots = \sum_{n=1}^{\infty} ax^{n-1}$$

for any constant a. The two expressions are equal for $|x| < 1$. The radius of convergence of the series is $R = 1$. For $x = 2$ the right side makes sense but the left does not.

Example: Expand $1/(4 + x^2)$ as a power series.

$$\frac{1}{4+x^2} = \frac{1/4}{1-(-x^2/4)} = \frac{1}{4}[1 - x^2/4 + x^4/16 - \ldots + (-1)^n x^{2n}/4^n + \ldots$$

Calculus and Power Series: If the power series $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence R where $0 < R \leq \infty$ then the function $f(x)$ so defined is differentiable on the interval $a - R < x < a + R$ and

$$\begin{aligned}
f'(x) &= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots = \sum_{n=1} nc_n(x-a)^{n-1} \\
\int f(x)\,dx &= C + c_0(x-a) + \frac{c_1}{2}(x-a)^2 + \frac{c_2}{3}(x-a)^3 + \frac{c_3}{4}(x-a)^4 + \ldots \\
&= C + \sum_{n=0} \frac{c_n}{n+1}(x-a)^{n+1}
\end{aligned}$$

Both these power series have the same radius of convergence R as does $\sum_n^{\infty} c_n(x-a)^n$

Example:

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}(1+x+x^2+x^3+x^4+\ldots) = 1+2x+3x^2+4x^3+\ldots = \sum_{n=1}^{\infty} nx^{n-1}$$

The series converges for $|x| < 1$ by the theorem. Similarly

$$\begin{aligned}
\ln(1-x) &= -\int \frac{dx}{1-x} = -\int (1 + x + x^2 + x^3 + x^4 + \ldots)\,dx \\
&= C + x + x^2/2 + x^3/3 + x^4/4 + \ldots = C + \sum_{n=0}^{\infty} x^{n+1}/(n+1)
\end{aligned}$$

and one easily checks by setting $x = 0$ that $C = 0$. The series converges for $|x| < 1$. One sees therefore that

$$\ln x = \ln[1 - (1-x)] = \sum_{n=0}^{\infty} (1-x)^{n+1}/(n+1) = \sum_{n=1}^{\infty} (1-x)^n/n$$

Example:

$$\tan^{-1}(x) = \int \frac{1}{1+x^2}\,dx = \int \sum_{n=0}^{\infty} (-x^2)^n\,dx = C + \sum_{n=0}^{\infty} (-1)^n x^{2n+1}/(2n+1)$$

Setting $x = 0$, we see that $C = 0$. The series converges for $|x^2| < 1$ which means $|x| < 1$.