10.7 Power Series: A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

where c_n ia a sequence and a a constant and this defines a function of x wherever it converges. For example

$$\sum_{n=0}^{\infty} a\, x^n = \frac{a}{1-x} \qquad \text{provided } |x| < 1$$

because this is a geometric series. This expansion shows us that the division $a/(1-x)$ can be approximated by a polynomial expression which means simply multiplications and divisions.

Example: Determine for which values of x the series converges.

$$\sum_{n=1}^{\infty} n 2^n (x-3)^n.$$

Solution: Try the ratio test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)2^{n+1}(x-3)^n}{n2^n} \right| = \lim_{n\to\infty} \frac{2(n+1)}{n} |x-3| = 2|x-3|$$

The series converges provided $2|x-3| < 1$ so that $2.5 < x < 3.5$ and it diverges if $2|x-3| > 1$. The *radius of convergence* for this series is $1/2$.

Theorem 3. For the power series $\sum_n^{\infty} c_n (x-a)^n$ there are only three possibilities

1. The power series converges at $x = a$ only and diverges for all $x \neq a$, or
2. The power series converges for all real x, or
3. There exists a constant $R > 0$ so that the power series converges for all x, $a - R < x < a + R$ and diverges for $|x - a| > R$

The R is called the radius of convergence. Picture

Proof. This is an application of the ratio test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n (x-a)^n} \right| = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| |x-a|$$

The limit exists for $x = a$ at least. If

$$\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|$$

does not exist or is infinity then that is the only x for which it converges. If however the above limit exists and it is L then the ratio test says the series converges provided $L|x-a| < 1$ and diverges is $L|x-a| > 1$ If $L \neq 0$ this is case 3 and $L = 1/R$; if $L = 0$ then this is case 2. $\square$

Example: For what values of x does $\sum_{n=0}^{\infty} \frac{3^n}{n!}(x+2)^{n+1}$ converge? What is the radius of convergence?

Solution. Apply the ratio test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \lim_{n\to\infty} \left| \frac{3^{n+1}(x+2)^{n+2}/(n+1)!}{3^n(x+2)^{n+1}/n!} \right| = \lim_{n\to\infty} \left| \frac{3(x+2)}{n+1} \right| = 0$$

This says that the series converges for all x by the ratio test. The radius of convergence is ∞.

Example: For what values of x does the series $\sum_{n=0}^{\infty} \frac{n}{5^n}(x+1)^n$ converge?

Solution. Apply the ratio test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{n+1}{5^{n+1}}(x+1)^{n+1}}{\frac{n}{5^n}(x+1)^n} \right| = \lim_{n\to\infty} \frac{n+1}{5n}|x+1| = |x+1|/5$$

So the interval of convergence is $|x+1|/5 < 1$ so that $-6 < x < 4$.