10.5 Alternating Series Test: An alternating series is one that in which the terms alternate sign.

Examples: $\sum^{\infty} \frac{(-1)^n}{n}$, $\sum^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$

Sometimes $\sum^{\infty} \frac{\cos(n\pi)}{n}$

Example; What are the first 6 partial sums of the sequence

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

Solution.

$$\begin{aligned}
s_1 = &\quad a_1 &= 1 \\
s_2 = &\quad a_1 + a_2 &= 1 - 1/2 = 1/2 \\
s_3 = &\quad s_2 + a_3 &= 1/2 + 1/3 = 5/6 \\
s_4 = &\quad s_3 + a_4 &= 5/6 - 1/4 = 7/12 \\
s_5 = &\quad s_4 + a_5 &= 7/12 + 1/5 = 47/60 \\
s_6 = &\quad s_5 + a_6 &= 47/60 - 1/6 = 37/60
\end{aligned}$$

Observe that the previous two partial sums s_{n-1} and s_n bracket the next s_{n+1}.

Alternating Series Test. Suppose $b_n \geq 0$ is a DECREASING sequence: $b_{n+1} \leq b_n$ and

$$\lim_{n\to\infty} b_n = 0$$

then the two series

$$\sum_{n=1}^{\infty} (-1)^n b_n = -\sum_{n=1}^{\infty} (-1)^{n-1} b_n$$

converge. Moreover the mth partial sum s_m approximates the final limiting sum:

$$|\sum_{n=1}^{\infty} (-1)^n b_n - \sum_{n=1}^{m} (-1)^n b_n| \leq b_{n+1}$$

Proof. Consider $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ to be specific. Compute

$$s_{n+2} - s_n = (-1)^n b_{n+1} + (-1)^{n+1} b_{n+2} = (-1)^n (b_{n+1} - b_{n+2})$$

In the case n is even, $(-1)^n = 1$ and this says $s_{n+2} \geq s_n$ Therefore $s_2, s_4, s_6, s_8, \ldots$ is increasing. $s_2, s_4, s_6, s_8, \ldots$. Conversely $s_1 = b_1, s_3, s_5, s_7, \ldots$ is a decreasing sequence. The latter sequence is bounded below by the former sequence and so they both converge. Note $s_2 = s_1 - b_2$ and $s_n = s_{n-1} - bn$ if n is even. Since b_n gets small the limits of the two sequences must be the same.

Example: $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

Solution. Converges.

Example: $\sum_{n=1}^{\infty}(-1)^{n-1}(e^{1/n}-1)$
Solution. Converges.
Example: $\sum_{n=1}^{\infty}(-1)^{n}n^{1/n}$.
Solution. Diverges.