10.4 Ratio and Root Test: Convergence for series with terms that change sign is delicate because it may depend on fortuitous cancellations as in the alternating series. When the convergence is not dependent on the fortuitous cancellations then it is called absolute convergence.

Definition. A series $\sum_n^\infty a_n$ converges absolutely if $\sum_n^\infty |a_n|$ converges.

Examples: $\displaystyle\sum^\infty \frac{(-1)^n}{n}, \sum^\infty \frac{(-1)^{n-1}}{n^{3/2}}$ both converge by the alternating series test but only the latter converges absolutely.

Theorem 1. An *absolutely* convergent series $\sum_n^\infty a_n$ is convergent.

Proof. Consider the series $\sum_n^\infty b_n$ where $b_n = a_n + |a_n|$. Then

$$0 \leq b_n \leq 2|a_n|$$

If $\sum_n^\infty a_n$ is absolutely convergent then $\sum_n^\infty b_n$ is convergent by the comparison test (comparing to $\sum 2|a_n|$.) On the other hand the partial sums are

$$\sum_i^n b_i = \sum_i^n a_i + \sum_i^n |a_i|$$

and we know the series on the left converges and the second series on the right also converges so that $\sum_i^n a_i$ must also converge. $\square$

Example; $\displaystyle\sum_n^\infty \frac{sinn}{2^n}$ converges absolutely and so it converges.

Note convergence does not imply absolute convergence; $\sum(-1)^{n-1}1/n$. A series which converges but not absolutely is called *conditionally convergent.*

Ratio Test. Suppose $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$

1. If $L < 1$ then $\sum_n^\infty a_n$ converges absolutely.
2. If $L > 1$ or $L = \infty$ then the series diverges.
3. If $L = 1$ or L does not exist them the test fails.

Root Test. Suppose $\displaystyle\lim_{n\to\infty} |a_n|^{1/n} = L$ exists. Then the same alternatives as above hold.

Proof of the Ratio Test. Consider the case $L < 1$. Choose r, $L < r < 1$. For N large enough $|a_{N+1}| \leq r|a_N|$, so that $|a_{N+2}| \leq r^2|a_N|$ so that

$$\sum_{i=N}^\infty |a_i| \leq |a_N| \sum_{i=N}^\infty r^{i-N}$$

The latter is the geometric series with $r < 1$ and $a = |a_N|$ so that we get convergence by comparison.

Example. Does the series $\displaystyle\sum_n^\infty \frac{2^n}{n!}$ converge absolutely, converge conditionally or diverge?

Solution. Here $a_n = 2^n/n!$. Apply the ratio text

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{2^{n+1}/(n+1)!}{2^n/n!} = \lim_{n\to\infty} \frac{2}{n+1} = 0$$

The series converges absolutely.

Example. Determine whether the series is abolutely convergent, conditionally convergent or divergent.

$$\sum_{n=4}^{\infty} \frac{n^4}{4^n}$$

Solution. The ratio test may work here because of the 4^n.

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)^4/4^{n+1}}{n^4} 4^n = \lim_{n\to\infty} \frac{1}{4} \frac{(n+1)^4}{n^4} = \frac{1}{4}.$$

Example. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n}$ converges conditionally but not absolutely.

Example: Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-10)^n}{n^n}$ converge absolutely, converge conditionally or diverge?

Solution. The root test is appropriate here.

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left| \frac{(-10)^n}{n^n} \right|^{1/n} = \lim_{n\to\infty} \frac{10}{n} = 0$$

so that the series converges absolutely.