8.3 Comparison Test:

Example: Does $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n-\sqrt{n}-1}$ converge or diverge. The terms are all larger than $1/n$ the harmonic series and so the series must diverge.

Comparison Test. Given a series $\sum_{n=1}^{\infty} a_n$ and $a_n \geq 0$ suppose it is possible to find a series $\sum_{n=1}^{\infty} b_n$ so that $b_n \geq 0$ and either

1. $a_n \leq b_n$ for all n and the series $\sum_{n=1}^{\infty} b_n$ converges. Then $\sum_{n=1}^{\infty} a_n$ converges also.

   or

2. $a_n \geq b_n$ for all n and the series $\sum_{n=1}^{\infty} b_n$ diverges. Then $\sum_{n=1}^{\infty} a_n$ diverges also

Explanation: Look at the area interpretation of the series. In the first case, the area for the b_n series is finite and larger than that for the a_n series and so the area for the latter series must also be finite. Similarly in the second case: the area for the b_n series is infinite but smaller than the area for the a_n series.

Example. $\displaystyle\sum_{n=3}^{\infty} \frac{5}{10^n+\sqrt{n}}$ converges by comparison with the geometric series $5\sum_{n=3}^{\infty} 10^{-n}$ which is convergent ($r = 10^{-1}$; $a = 5 \cdot 10^{-3}$.)

Example. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges by comparison with the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ because $1/n \leq 1/\ln n$.

Limit Comparison Test: Given a series $\sum_{n}^{\infty} a_n$ where $a_n \geq 0$, suppose there is a series $\sum_{n}^{\infty} b_n$ where $b_n \geq 0$ so that

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ exists then the two series either both converge or both diverge.

2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum^{\infty} b_n$ converges implies $\sum^{\infty} a_n$ converges.

3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum^{\infty} b_n$ diverges then so does $\sum^{\infty} a_n$.

Example. Determine whether the series $\sum_{n=2}^{\infty} \frac{n+2}{n^2+n+1}$ converges or diverges.
Solution. Compare to the harmonic series $\sum^{\infty} 1/n$. Then

$$a_n = \frac{n+2}{n^2+n+1} \qquad b_n = 1/n$$

so that

$$\frac{a_n}{b_n} = \frac{n^2+2n}{n^2+n+1} \qquad \text{and} \qquad \lim_{n\to\infty} \frac{a_n}{b_n} = 1$$

Therefore the two series either both converge or both diverge. Here, since the harmonic series diverges, so does the original series.

Example:Determine whether the series $\sum_{n=2}^{\infty} \frac{n}{(n-1)^{5/2}}$ converges or diverges.

Example. Compare to $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \frac{\frac{n}{(n-1)^{5/2}}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \left(\frac{n}{(n-1)}\right)^{5/2} = 1$$

The series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$ converges $(p > 1)$. Therefore the original series converges.

Note that the series

$$\sum_{n=2}^{\infty} \frac{n}{(n-1)^{5/2}} \geq \sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$$

and so the first comparison test doesn't work.