**8.3 Integral Test:**

Series cannot always be summed exactly. Primarily we will be interested in convergence or divergence.

Series as Area. If $a_n \geq 0$ then the partial sum $s_n = \sum_{i=1}^{n} a_i$ represents the area indicated below

Now suppose that $a_n = f(n)$ where $f(x) \geq 0$ and $f(x)$ is decreasing. From the picture it is clear that

$$\int_0^n f(x)\,dx \geq \sum_{i=1}^{n} a_n \geq \int_1^{n+1} f(x)\,dx$$

Integral Test. Suppose there is a continuous, positive decreasing function $f(x)$ and $a_n = f(n)$. Then

$$\int_1^\infty f(x)dx \text{is convergent if and only if} \sum_{n=1}^{\infty} a_n \text{is convergent}$$

Example: The $p$ series:

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if

$$\int_1^\infty \frac{1}{x^p}\,dx$$

converges. Recall that the latter converges if and only if $p > 1$ because

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t x^{-p}\,dx = \lim_{t\to\infty} \frac{t^{1-p}}{1-p} - \frac{1}{1-p}$$

provided $p \neq 1$. The limit on the right exists if and only if $p > 1$. If $p = 1$ then we get a $\ln t$ which does not converge either. So the integral converges if and only if $p > 1$. By the integral test

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if and only if } p > 1.$$

Example: $\sum_{n=5}^{\infty} 1/\sqrt{n}$ diverges

Example: The *harmonic* series $\sum_{n=1}^{\infty} 1/n$ diverges

Example: $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ converges because

$$\int_0^\infty \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \tan^{-1} t = \pi/2.$$